SPARSE DOMINATION AND WEIGHTED ESTIMATES FOR ROUGH BILINEAR SINGULAR INTEGRALS

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Abstract. Let $r > \frac{4}{3}$ and let $\Omega \in L^r(\mathbb{S}^{2n-1})$ have vanishing integral. We show that the bilinear rough singular integral

$$T_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^{2n}} f(x-y)g(x-z)\, dydz,$$

satisfies a sparse bound by (p,p,p)-averages, where p is bigger than a certain number explicitly related to r and n. As a consequence we deduce certain quantitative weighted estimates for bilinear homogeneous singular integrals associated with rough homogeneous kernels.

1. Introduction

In 1952, Calderón and Zygmund [3] established the existence and $L^p(\mathbb{R}^n)$ boundedness of the following rough singular integrals

$$T_K(f)(x_1, x_2, \dots, x_n) = \int_{\mathbb{R}^n} f(s_1, \dots, s_n) K(x_1 - s_1, \dots, x_n - s_n) ds_1 \cdots ds_n,$$

where f is an integrable function defined on $\mathbb{R}^n$ and

$$K(x_1, \dots, x_n) = \rho^{-n}\Omega(\alpha_1, \dots, \alpha_n),$$

with $x_j = \rho \cos\alpha_j$ for all j, $\rho > 0$, and $\alpha_1, \alpha_2, \dots, \alpha_n$ are the direction angles of $(x_1, x_2, \dots, x_n)$. Later on, using the method of rotations, Calderón and Zygmund [4] proved that the operator

$$T_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y) dy$$

is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ whenever $\Omega \in L^1(\mathbb{S}^{n-1})$, $\int_{\mathbb{S}^{n-1}} \Omega\, d\sigma = 0$ and if the even part of Ω belongs to the class $L \log L(\mathbb{S}^{n-1})$.

Since 1956 this area has flourished and has been enriched by a considerable amount of work, which could not be all listed here. We note however the work of Christ [5], Christ and Rubio de Francia [6], Seeger [34], Tao [35], Duoandikoetxea and Rubio de

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Francia [14], Grafakos and Stefanov [16] among many others. The weighted theory of T_Ω is also quite rich; here we note the work of Duoandikoetxea [13] and Vargas [36] and we would like to direct attention to the recent works of [12, 32, 33].

In order to state more known results, we first introduce some notation. A collection $\mathcal{S}$ of cubes in $\mathbb{R}^n$ is called η-sparse if for each $Q \in \mathcal{S}$ there is $E_Q \subset Q$ such that $|E_Q| \geq \eta|Q|$, and such that $E_Q \cap E_{Q'} = \varnothing$ when $Q \neq Q'$ (here $0 < \eta < 1$). For an η-sparse collection of cubes $\mathcal{S}$ we use the notation

$$\mathsf{PSF}_{\mathcal{S};p_1,p_2}(f_1, f_2) := \sum_{Q\in\mathcal{S}} |Q|\langle f_1\rangle_{p_1,Q}\langle f_2\rangle_{p_2,Q}, \quad \langle f\rangle_{p,Q} := |Q|^{-\frac{1}{p}} \|f\mathbf{1}_Q\|_{L^p}.$$

It is known that the L^1 norm of the bilinear maximal operator plays an important role in the study of the forms PSF. We refer the readers to [9, 26, 28] for more details. Such expressions dominate quantities $|\langle T(f_1), f_2\rangle|$ for linear operators T. This type of domination is called sparse and plays an important role and finds wide applicability in harmonic analysis. For instance, it was used in the proof of A_2 conjecture [23, 24]. Earlier works related to sparse domination can be found in [2, 22, 23, 25, 30, 37] and the references therein. In 2017, Conde-Alonso et al. [8] obtained the following sparse domination for T_Ω:

$$|\langle T_\Omega(f_1), f_2\rangle| \leq \frac{Cp}{p-1} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,p}(f_1, f_2) \begin{cases} \|\Omega\|_{L^{r,1}\log L(\mathbb{S}^{d-1})}, & 1 < r < \infty, p \geq r'; \\ \|\Omega\|_{L^\infty(\mathbb{S}^{d-1})}, & 1 < p < \infty. \end{cases}$$

As a consequence, the authors in [8] deduced a new sharp quantitative A_p-weighted estimate for T_Ω. Subsequently, for all $\epsilon > 0$, Di Plinio, Hytönen, and Li [11], provided a sparse bound by $(1+\epsilon, 1+\epsilon)$-averages with linear growth in ϵ^{-1} for the associated maximal truncated singular integrals T_*, i.e., $\|T_*\|_{(1+\epsilon,1+\epsilon),sparse} \leq C\epsilon^{-1}$. As a corollary, certain novel quantitative weighted norm estimates were given for T_*.

The study of bilinear singular integrals originated in the celebrated work of Coifman and Meyer [7]. The main object of study is the bilinear operator (which is denoted as in the linear case without risk of confusion as its linear counterpart will not appear in the sequel)

$$T_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^{2n}} f(x-y)g(x-z)\, dydz, \tag{1.1}$$

where Ω is an integrable function on $\mathbb{S}^{2n-1}$ with mean value zero. The boundedness of rough bilinear singular integrals can be derived from uniform bounds for the bilinear Hilbert transforms (see [17], [15] for details). Let $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. In 2015, Grafakos, He and Honzík [17] obtained the $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ boundedness for T_Ω when $\Omega \in L^\infty(\mathbb{S}^{2n-1})$. Additionally, these authors showed that T_Ω is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if $\Omega \in L^r(\mathbb{S}^{2n-1})$ for $r \geq 2$. In 2018, Grafakos, He, and Slavíková [19] gave a criterion for $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ boundedness for certain bilinear operators. As an application, these authors improved the results in [17] as follows:

Theorem A. ([19]) Let $r > 4/3$ and $\Omega \in L^r(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$. Then $\|T_\Omega\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty$ whenever $2 \le p_1, p_2 \le \infty$, $1 \le p \le 2$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

For Ω in $L^r(\mathbb{S}^{2n-1})$, it is natural to ask for the exact range of (p_1, p_2, p) such that T_Ω maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. This problem is quite delicate. A counterexample of Grafakos, He and Slavíková [18] shows that there exists an Ω in $L^r(\mathbb{S}^{2n-1})$, $1 \le r < \infty$, which satisfies the Hörmander kernel condition on $\mathbb{R}^{2n}$, such that the associated T_Ω is unbounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \le p_1, p_2 \le \infty$ and $\frac{1}{p} + \frac{2n-1}{r} > 2n$. However, it is unknown whether T_Ω is bounded when the last condition fails.

In this work, we focus on the sparse domination of T_Ω for rough functions Ω. Note that the authors in [10] established a uniform domination of the family of trilinear multiplier forms with singularity over an one-dimensional subspace. Later Barron [1] considered the sparse domination for rough bilinear singular integrals with Ω in $L^\infty(\mathbb{S}^{2n-1})$.

Theorem B. ([1]) Suppose T_Ω is the rough bilinear singular integral operator defined by (1.1), with $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ and $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$. Then for any $1 < p < \infty$, there is a constant $C_{p,n} > 0$ so that

$$|\langle T_\Omega(f_1, f_2), f_3\rangle| \le C_{p,n} \|\Omega\|_{L^\infty(\mathbb{S}^{2n-1})} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S}}^{(p,p,p)}(f_1, f_2, f_3),$$

where the sparse (p_1, p_2, p_3)-averaging form is defined as

$$\mathsf{PSF}_{\mathcal{S}}^{(p_1,p_2,p_3)}(f_1, f_2, f_3) := \sum_{Q\in\mathcal{S}} |Q| \prod_{i=1}^{3} \langle f_i\rangle_{p_i,Q}, \text{ for } \ 1 \le p_i < \infty, \ i = 1, 2, 3.$$

In this paper, we establish sparse domination for bilinear rough operator T_Ω with $\Omega \in L^r(\mathbb{S}^{2n-1})$ for $r < \infty$. These Ω produce rougher singular integrals than the ones previously studied. As a result we deduce certain quantitative weighted estimates for rough bilinear singular integral operators. The main result of this paper is as follows:

Theorem 1.1. *Let $\Omega \in L^r(\mathbb{S}^{2n-1})$, $r > 4/3$, and $\int_{\mathbb{S}^{2n-1}} \Omega = 0$. Let T_Ω be the rough bilinear singular integral operator defined in* (1.1). *Then for $p > \max\{\frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\}$ there exists a constant $C = C_{p,n,r}$ such that*

$$|\langle T_\Omega(f_1, f_2), f_3\rangle| \le C \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S}}^{(p,p,p)}(f_1, f_2, f_3).$$

Remark 1.1. Letting $r \to \infty$, the restriction on p in Theorem 1.1 becomes $p > 1$ for $\Omega \in L^\infty(\mathbb{S}^{2n-1})$. Thus Theorem 1.1 coincides with the sparse domination result of Theorem B when $r = \infty$. Thus our work essentially extends that of [1] and all the weighted results it implies. Whether there is an explicit dependence of $C_{p,n,r}$ in Theorem 1.1 on p, even in the limiting case $r = \infty$, is still an interesting open problem.

In order to state our corollaries, we recall some background and introduce notation relevant to certain classes of weights. Let $p' = p/(p-1)$ be the dual exponent of p.

We recall the definition of the A_p weight classes: We say $w \in A_p$ for $1 < p < \infty$ if $w > 0$, $w \in L^1_{loc}$ and

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|} \int_Q w \Big) \Big(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \Big)^{p-1} < \infty.$$

In 2002 Grafakos and Torres [21] initiated the weighted theory for the multilinear singular operators but it was not until 2009 that Lerner et. al. [29] introduced the canonical Muckenhoupt vector A_p weight class, denoted by $A_{\vec{p}}$, which provides a natural analogue of the linear theory.

Definition 1.2 (**Multiple weight class** $A_{\vec{p}}$, [29])**.** Let $1 \le p_1, \ldots, p_m < \infty$, $\vec{w} = (w_1, \ldots, w_m)$, where w_i $(i = 1, \ldots, m)$ are nonnegative functions defined on $\mathbb{R}^n$, and denote $v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$. We say $\vec{w} \in A_{\vec{p}}$ if

$$[\vec{w}]_{A_{\vec{p}}} = \sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{w}}(t)dt \right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q w_i^{1-p_i'}(t)dt \right)^{\frac{1}{p_i'}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and the term $\left(\frac{1}{|Q|} \int_Q w_i^{1-p_i'}(t)dt \right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q w_i)^{-1}$ when $p_i = 1$.

More general weights class than $A_{\vec{p}}$ has also been considered by Li, Martell, and Ombrosi in [31]. For $m \ge 2$, given $\vec{p} = (p_1, \ldots, p_m)$ with $1 \le p_1, \ldots, p_m < \infty$ and $\vec{r} = (r_1, \ldots, r_{m+1})$ with $1 \le r_1, \ldots, r_{m+1} < \infty$, we say that $\vec{r} \prec \vec{p}$ whenever

$$r_i < p_i,\ i = 1, \ldots, m \text{ and } r'_{m+1} > p, \text{ where } \frac{1}{p} := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Definition 1.3 ($A_{\vec{p},\vec{r}}$ **weight class**, [31])**.** Let $m \ge 2$ be an integer, $\vec{p} = (p_1, \ldots, p_m)$ with $1 \le p_1, \ldots, p_m < \infty$ and $\vec{r} = (r_1, \ldots, r_{m+1})$ with $1 \le r_1, \ldots, r_{m+1} < \infty$. $1/p = \sum_{k=1}^m 1/p_k$. For each $w_k > 0$, $w_k \in L^1_{loc}$, set

$$w = \prod_{k=1}^m w_k^{p/p_k}.$$

We say that $\vec{w} = (w_1, ..., w_m) \in A_{\vec{p},\vec{r}}$ if $0 < w_i < \infty$, $1 \le i \le m$ and $[w]_{A_{\vec{p},\vec{r}}} < \infty$ with

$$[\vec{w}]_{A_{\vec{p},\vec{r}}} = \sup_Q \Big(\frac{1}{|Q|} \int_Q w(x)^{\frac{r'_{m+1}}{r'_{m+1}-p}} \,\mathrm{d}x \Big)^{1/p - 1/r'_{m+1}} \prod_{k=1}^m \Big(\frac{1}{|Q|} \int_Q w_k(x)^{-\frac{1}{\frac{p_k}{r_k}-1}} \,\mathrm{d}x \Big)^{1/r_k - 1/p_k}.$$

When $r_{m+1} = 1$ the term corresponding to w needs to be replaced by $(\frac{1}{|Q|} \int_Q w dx)^{\frac{1}{p}}$. Here and afterwards, the expression

$$\Big(\frac{1}{|Q|} \int_Q w_k(x)^{-\frac{1}{\frac{p_k}{r_k}-1}} \,\mathrm{d}x \Big)^{1/r_k - 1/p_k}$$

is understood as $\operatorname{esssup}_Q w_k^{-1/p_k}$ when $p_k = r_k$.

When $r_1 = \cdots = r_m = 1$, $A_{\vec{p},\vec{r}}$ coincides with the weight class $A_{\vec{p}}$ introduced by Lerner et al. [29]

As an application of the sparse domination, we obtain certain weighted estimates for T_Ω. The first result is concerned with multiple weights while the other with the one-weight case.

Corollary 1.2. *Let $\Omega \in L^r(\mathbb{S}^{2n-1})$ with $r > 4/3$ and $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$. Let $\vec{q} = (q_1, q_2)$, $\vec{p} = (p_1, p_2, p_3)$ with $\vec{p} \prec \vec{q}$ and $p_i > \max\{\frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\}$, $i = 1, 2, 3$. Let*

$$\mu_{\vec{v}} = \prod_{k=1}^{2} v_k^{q/q_k}$$

and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $1 < q < \max\{\frac{24n+3r-4}{16n}, \frac{24n+r}{16n}\}$ and let $q_3 = q'$. Then there is a constant $C = C_{\vec{p},\vec{q},r,n}$ such that

$$\|T_\Omega(f, g)\|_{L^q(\mu_{\vec{v}})} \le C \|\Omega\|_{L^r} [\vec{v}]_{A_{\vec{q},\vec{p}}}^{\max_{1\le i\le 3}\{\frac{p_i}{q_i - p_i}\}} \|f\|_{L^{q_1}(v_1)} \|g\|_{L^{q_2}(v_2)}.$$

Corollary 1.3. *Let $\Omega \in L^r(\mathbb{S}^{2n-1})$ with $r > 4/3$ and $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$. For $w \in A_{p/2}$, $\max\{2, \frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\} < p < \max\{\frac{24n+3r-4}{8n}, \frac{24n+r}{8n}\}$, there exists a constant $C = C_{w,p,n,r}$ such that*

$$\|T_\Omega(f_1, f_2)\|_{L^{p/2}(w)} \le C \|\Omega\|_{L^r} \|f_1\|_{L^p(w)} \|f_2\|_{L^p(w)}.$$

Remark 1.4. We make few comments about Corollaries 1.2 and 1.3.

- The class of weights in Corollary 1.2 is slightly different than that used in [1].
- In Theorem A there is a restriction $p_i > 2$. It is interesting that in Corollary 1.2, when $\frac{4}{3} < r < 8n$ it is easy to see that $p_i > 2$, $i = 1, 2$. However, when $r \ge 8n$, then p_1, p_2 could be smaller than 2. This means that, in some sense, q_i enjoys more freedom in Corollary 1.2, since we only require $q > 1$ and there is no need to assume that each $q_i > 2$.
- We guess that the index regions in the above two corollaries are far from optimal. To find the best region for the above weighted results should be a very interesting problem.

The main idea in the proof of Theorem 1.1 is to elaborate on the decomposition [14] for the rough kernel into smooth kernels with controlled (summable) growth of constants. Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $2 \le p_1, p_2 \le \infty$ and $1 \le p \le 2$. If $\Omega \in L^r(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$, $4/3 < r \le \infty$, for $j > 0$ and $0 < \delta < \frac{1}{r'}$, Grafakos, He, and Honzik [17] showed that $\|T_j\|_{L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \le C \|\Omega\|_{L^r} 2^{(2n-\delta)j}$. Obviously, there is no appropriate decay on the right side of this inequality. In the proof of Theorem 1.1, we need to sum over all $j \in \mathbb{Z}$. Therefore, this inequality is not sufficient for our purpose. In this paper, we will handle the decay in j for norm estimate of T_j with $j > 0$ by adapting the tensor-type wavelet decompositions techniques from [19] in order to prove the sparse bound for T_Ω.

The article is organized as follows. Section 2 contains definitions and basic lemmas. An analysis of the Calderón-Zygmund kernel is given in Section 3. Section 4 and Section 5 are devoted to the demonstration of the proof of Theorem 1.1 and its corollaries. Throughout this paper, the notation $\lesssim$ will be used to denote an inequality with an inessential constant on the right. We denote by $\ell(Q)$ the side length of a cube Q in $\mathbb{R}^n$ and by $\operatorname{diam}(Q)$ its diameter. For $\lambda > 0$ we use the notation λQ for the cube with the same center as Q and side length $\lambda\ell(Q)$.

2. Definitions and main lemmas

In this section we consider a general bilinear operator that commutes with translations

$$T[K](f_1, f_2)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - x_1, x - x_2) f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \tag{2.1}$$

and assume it is a bounded bilinear operator mapping $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n) \to L^{\alpha}(\mathbb{R}^n)$ for some $r_1, r_2, \alpha \geq 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{\alpha}$. It is assumed that the kernel K of $T[K]$ has a decomposition of the form

$$K(u, v) = \sum_{s\in\mathbb{Z}} K_s(u, v), \tag{2.2}$$

where K_s is a smooth truncation of K that enjoys the property

$$\operatorname{supp} K_s \subset \big\{(u, v) \in \mathbb{R}^{2n} : \ 2^{s-2} < |u| < 2^s,\ 2^{s-2} < |v| < 2^s\big\}.$$

The truncation of $T[K]$ is defined as

$$T[K]_{t_1}^{t_2}(f_1, f_2)(x) := \sum_{t_1<s<t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_s(x - x_1, x - x_2) f_1(x_1) f_2(x_2)\, dx_1 dx_2, \tag{2.3}$$

where $0 < t_1 < t_2 < \infty$. See Section 2.1 in [1] for remarks on this type of truncated operators. In this work, we assume that the truncated norm satisfies

$$\sup_{0<t_1<t_2<\infty} \|T[K]_{t_1}^{t_2}\|_{L^{r_1}\times L^{r_2}\to L^{\alpha}} < \infty, \tag{2.4}$$

for some $r_1, r_2, \alpha \geq 1$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{\alpha}$. To study bilinear operators T, we often work with the trilinear form of the type $\langle T(f_1, f_2), f_3\rangle = \int_{\mathbb{R}^n} T(f_1, f_2) f_3(x)\, dx$. In our case, the trilinear truncated form is

$$\langle T[K]_{t_1}^{t_2}(f_1, f_2), f_3\rangle = \int_{\mathbb{R}^n} T[K]_{t_1}^{t_2}(f_1, f_2) f_3\, dx.$$

Denoting by $C_T(r_1, r_2, \alpha)$ the following constant

$$C_T(r_1, r_2, \alpha) := \sup_{0<t_1<t_2<\infty} \frac{\big|\langle T[K]_{t_1}^{t_2}(f_1, f_2), f_3\rangle\big|}{\|f_1\|_{L^{r_1}} \|f_2\|_{L^{r_2}} \|f_3\|_{L^{\alpha'}}}\,, \tag{2.5}$$

then (2.4) is equivalent to $C_T(r_1, r_2, \alpha) < \infty$.

Remark 2.1. If a bilinear operator of the form (2.1) is bounded from $L^{r_1} \times L^{r_2} \to L^{\alpha}$ with $\alpha \geq 1$, then so do all of its smooth truncations with kernels

$$K(u, v) G(u/2^t) G(v/2^{t'})$$

uniformly on t, t'. Here G is any function whose Fourier transform is integrable.

To see this, we express (2.1) in multiplier form as follows

$$\int_{\mathbb{R}^{2n}} \widehat{G}(\xi_1', \xi_2') \left[\int_{\mathbb{R}^{2n}} \widehat{K}(\xi_1 - \xi_1', \xi_2 - \xi_2') \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right] d\xi_1' \, d\xi_2'$$

and then we pass the $L^\alpha(dx)$ norm on the square bracket.

Definition 2.2 (**Stopping collection** [8])**.** Let $\mathcal{D}$ be a fixed dyadic lattice in $\mathbb{R}^n$ and $Q \in \mathcal{D}$ be a fixed dyadic cube in $\mathbb{R}^n$. A collection $\mathcal{Q} \subset \mathcal{D}$ of dyadic cubes is a *stopping collection* with top Q if the elements of $\mathcal{Q}$ satisfy

$$L, L' \in \mathcal{Q}, L \cap L' \neq \emptyset \Rightarrow L = L'$$

$$L \in \mathcal{Q} \Rightarrow L \subset 3Q,$$

and enjoy the separation properties

(i) if $L, L' \in \mathcal{Q}$, $|s_L - s_{L'}| \geq 8$, then $7L \cap 7L' = \emptyset$.

(ii) $\displaystyle\bigcup_{\substack{L \in \mathcal{Q} \\ 3L \cap 2Q \neq \emptyset}} 9L \subset \bigcup_{L \in \mathcal{Q}} L =: sh\mathcal{Q}$.

Here $s_L = \log_2 \ell(L)$, where $\ell(L)$ is the length of the cube L.

Let $\mathbf{1}_A$ be the characteristic function of a set A. We use M_p to denote the power version of the Hardy-Littlewood maximal function

$$M_p(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(y)|^p dy \right)^{\frac{1}{p}},$$

where the supremum is taken over cubes $Q \subset \mathbb{R}^n$ containing x.

We need the following definition.

Definition 2.3 ($\mathcal{Y}_p(\mathcal{Q})$ **norm**, [8])**.** Let $1 \leq p \leq \infty$ and let $\mathcal{Y}_p(\mathcal{Q})$ be the subspace of $L^p(\mathbb{R}^n)$ of functions satisfying supp $h \subset 3Q$ and

$$\infty > \|h\|_{\mathcal{Y}_p(\mathcal{Q})} := \begin{cases} \max\left\{ \|h \mathbf{1}_{\mathbb{R}^n \setminus sh\mathcal{Q}}\|_\infty, \sup_{L \in \mathcal{Q}} \inf_{x \in \widehat{L}} M_p h(x) \right\}, & p < \infty, \\ \|h\|_\infty, & p = \infty, \end{cases} \tag{2.6}$$

where $\hat{L}$ is the (nondyadic) 2^5-fold dilation of L. We also denote by $\mathcal{X}_p(\mathcal{Q})$ the subspace of $\mathcal{Y}_p(\mathcal{Q})$ of functions satisfying

$$b = \sum_{L \in \mathcal{Q}} b_L, \quad \text{supp } b_L \subset L.$$

Furthermore, we say $b \in \dot{\mathcal{X}}_p(\mathcal{Q})$ if

$$b \in \mathcal{X}_p(\mathcal{Q}), \quad \int_L b_L = 0, \quad \forall L \in \mathcal{Q}.$$

$\|b\|_{\mathcal{X}_p(\mathcal{Q})}$ denotes $\|b\|_{\mathcal{Y}_p(\mathcal{Q})}$ when $b \in \mathcal{X}_p(\mathcal{Q})$ and similar notation for $b \in \dot{\mathcal{X}}_p(\mathcal{Q})$. We may omit $\mathcal{Q}$ and simply write $\| \cdot \|_{\mathcal{X}_p}$ or $\| \cdot \|_{\mathcal{Y}_p}$.

Let $a \wedge b$ denote the minimum of two real numbers a and b. Given a stopping collection $\mathcal{Q}$ with top cube Q, we define

$$\Lambda_{\mathcal{Q}}{}_{t_1}^{t_2}(f_1, f_2, f_3) = \frac{1}{|Q|}\Big[\langle T[K]_{t_1}^{t_2\wedge s_Q}(f_1\mathbf{1}_Q, f_2), f_3\rangle - \sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}}\langle T[K]_{t_1}^{t_2\wedge s_L}(f_1\mathbf{1}_L, f_2), f_3\rangle\Big]. \tag{2.7}$$

Then the support condition

$$\mathrm{supp}K_s \subset \big\{(u,v)\in\mathbb{R}^{2n}:\ 2^{s-2}<|u|<2^s,\, 2^{s-2}<|x_2|<2^s\big\}.$$

gives that

$$\Lambda_{\mathcal{Q}}{}_{t_1}^{t_2}(f_1, f_2, f_3) = \Lambda_{\mathcal{Q}}{}_{t_1}^{t_2}(f_1\mathbf{1}_Q, f_2 I_{3Q}, f_3\mathbf{1}_{3Q}).$$

For simplicity, we will often suppress the dependence of $\Lambda_{\mathcal{Q}}{}_{t_1}^{t_2}$ on t_1 and t_2 by writing $\Lambda_{\mathcal{Q}}(f_1, f_2, f_3) = \Lambda_{\mathcal{Q}}{}_{t_1}^{t_2}(f_1, f_2, f_3)$, when there is no confusion.

Lemma 2.1 ([1]). *Let T be a bilinear operator with kernel K as the above, such that K can be decomposed as in* (2.2) *and suppose that the constant C_T defined in* (2.5) *satisfies*

$$C_T = C_T(r_1, r_2, \alpha) < \infty$$

for some $1 \le r_1, r_2, \alpha < \infty$ with $1/r_1 + 1/r_2 = 1/\alpha$. Assume that there exist indices $1 \le p_1, p_2, p_3 \le \infty$ and a positive constant C_L such that for all finite truncations, all dyadic lattices $\mathcal{D}$, and all stopping collections $\mathcal{Q}$ with top cube Q, the quantity $\Lambda_{\mathcal{Q}}{}_{\mu}^{\nu}(f_1, f_2, f_3)$ satisfies uniformly for all $\mu < \nu$:

$$\begin{aligned}
\Lambda_{\mathcal{Q}}{}_{\mu}^{\nu}(b, g_2, g_3) &\le C_L|Q|\|b\|_{\dot{\mathcal{X}}_{p_1}}\|g_2\|_{\mathcal{Y}_{p_2}}\|g_3\|_{\mathcal{Y}_{p_3}};\\
\Lambda_{\mathcal{Q}}{}_{\mu}^{\nu}(g_1, b, g_3) &\le C_L|Q|\|g_1\|_{\mathcal{Y}_\infty}\|b\|_{\dot{\mathcal{X}}_{p_2}}\|g_3\|_{\mathcal{Y}_{p_3}};\\
\Lambda_{\mathcal{Q}}{}_{\mu}^{\nu}(g_1, g_2, b) &\le C_L|Q|\|g_1\|_{\mathcal{Y}_\infty}\|g_2\|_{\mathcal{Y}_\infty}\|b\|_{\dot{\mathcal{X}}_{p_3}}.
\end{aligned} \tag{2.8}$$

Then there is a constant c_n depending only on the dimension n such that the quantity $\Lambda_{\mu}^{\nu}(f_1, f_2, f_3) = \langle T[K]_{\mu}^{\nu}(f_1, f_2), f_3\rangle$ satisfies

$$\sup_{0<\mu<\nu<\infty}|\Lambda_{\mu}^{\nu}(f_1, f_2, f_3)| \le c_n[C_T + C_L]\sup_{\mathcal{S}}\mathsf{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3)$$

for all $f_j \in L^{p_j}(\mathbb{R}^n)$ with compact support, where $\vec{p} = (p_1, p_2, p_3)$ and the supremum on the right is taken with respect to all sparse collections $\mathcal{S}$.

Lemma 2.1 is a crucial ingredient of our proof as it implies that

$$|\langle T_\Omega(f_1, f_2), f_3\rangle| \le (C_T + C_L)\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sup_{\mathcal{S}}\mathsf{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3),$$

where $\vec{p} = (p_1, p_2, p_3)$.

Next we will consider the interpolation involving $\mathcal{Y}_q$-spaces, of which the precursor can be seen in [11, Proposition 2.1]. We only give the particular cases which we need to prove Theorem 1.1, however, more general results are available.

Lemma 2.2. *Let $0 < A_2 \le A_1 < \infty$, $0 < \epsilon < 1$, and $q = 1 + 2\epsilon$. Suppose that $\Lambda_{\mathcal{Q}}$ is a (sub)-trilinear form such that*

$$|\Lambda_{\mathcal{Q}}(b, f, g)| \lesssim A_1 \|b\|_{\dot{\mathcal{X}}_1} \|f\|_{\mathcal{Y}_1} \|g\|_{\mathcal{Y}_1}, \tag{2.9}$$

$$|\Lambda_{\mathcal{Q}}(b, f, g)| \lesssim A_2 \|b\|_{\dot{\mathcal{X}}_3} \|f\|_{\mathcal{Y}_3} \|g\|_{\mathcal{Y}_3}. \tag{2.10}$$

Then we have

$$|\Lambda_{\mathcal{Q}}(f_1, f_2, f_3)| \lesssim A_1^{1-\epsilon} A_2^{\epsilon} \|f_1\|_{\dot{\mathcal{X}}_q} \|f_2\|_{\mathcal{Y}_q} \|f_3\|_{\mathcal{Y}_q}.$$

Proof. Without loss of generality, we may assume $A_2 \le A_1 = 1$, and $\|f_1\|_{\dot{\mathcal{X}}_q} = \|f_2\|_{\mathcal{Y}_q} = \|f_3\|_{\mathcal{Y}_q} = 1$, then it is enough to prove $\Lambda_{\mathcal{Q}}(f_1, f_2, f_3) \lesssim A_2^{\epsilon}$.

Fix $\lambda \ge 1$ and denote $f_{>\lambda} = f \mathbf{1}_{|f|>\lambda}$. We decompose $f_1 = b_1 + g_1$, where

$$b_1 := \sum_{L \in \mathcal{Q}} \big((f_1)_{>\lambda} - \frac{1}{|L|} \int_L (f_1)_{>\lambda} \big) \mathbf{1}_L.$$

For f_2 and f_3, we decompose $f_2 = b_2 + g_2$, $f_3 = b_3 + g_3$, where $b_i := (f_i)_{>\lambda}$, $i = 2, 3$. Then it holds that

$$\begin{aligned} &\|b_1\|_{\dot{\mathcal{X}}_1} \lesssim \lambda^{1-q}, \quad \|g_1\|_{\dot{\mathcal{X}}_1} \le \|g_1\|_{\dot{\mathcal{X}}_3} \lesssim \lambda^{1-\frac{q}{3}}, \\ &\|b_2\|_{\mathcal{Y}_1} \lesssim \lambda^{1-q}, \quad \|g_2\|_{\mathcal{Y}_1} \le \|g_2\|_{\mathcal{Y}_3} \lesssim \lambda^{1-\frac{q}{3}}, \\ &\|b_3\|_{\mathcal{Y}_1} \lesssim \lambda^{1-q}, \quad \|g_3\|_{\mathcal{Y}_1} \le \|g_3\|_{\mathcal{Y}_3} \lesssim \lambda^{1-\frac{q}{3}}. \end{aligned} \tag{2.11}$$

The proofs of these estimates are given at the end of this lemma. Now we estimate $|\Lambda_{\mathcal{Q}}(f_1, f_2, f_3)|$ by the sum of the following eight terms

$$\begin{aligned} &|\Lambda_{\mathcal{Q}}(b_1, b_2, b_3)| + |\Lambda_{\mathcal{Q}}(g_1, b_2, b_3)| + |\Lambda_{\mathcal{Q}}(b_1, g_2, b_3)| + |\Lambda_{\mathcal{Q}}(b_1, b_2, g_3)| \\ &\quad + |\Lambda_{\mathcal{Q}}(g_1, g_2, b_3)| + |\Lambda_{\mathcal{Q}}(g_1, b_2, g_3)| + |\Lambda_{\mathcal{Q}}(b_1, g_2, g_3)| + |\Lambda_{\mathcal{Q}}(g_1, g_2, g_3)|. \end{aligned}$$

For the last term we use assumption (2.10) while we use (2.9) to estimate the remaining seven terms. It follows that

$$|\Lambda_{\mathcal{Q}}(f_1, f_2, f_3)| \lesssim \lambda^{3-3q} + 3\lambda^{2-2q} + 3\lambda^{1-q} + A_2 \lambda^{3-q}.$$

Noting that $1 - q = -2\epsilon$ and $\lambda \ge 1$, then we have

$$\begin{aligned} |\Lambda_{\mathcal{Q}}(f_1, f_2, f_3)| &\lesssim 3\lambda^{-2\epsilon} + 3\lambda^{-4\epsilon} + \lambda^{-6\epsilon} + A_2 \lambda^{3-q} \\ &\lesssim 7\lambda^{-2\epsilon} + A_2 \lambda^{2-2\epsilon} \\ &\lesssim \lambda^{-2\epsilon}(7 + A_2 \lambda^2). \end{aligned} \tag{2.12}$$

Let $\lambda = A_2^{-\frac{1}{2}}$, then $|\Lambda_{\mathcal{Q}}(f_1, f_2, f_3)| \lesssim A_2^{\epsilon}$.

It remains to derive estimates (2.11) for b_i and g_i. We only demonstrate how to compute $\|g_1\|_{\mathcal{Y}_2} \lesssim \lambda^{1-\frac{q}{3}}$ as the estimates for b_1, b_2, b_3, g_2, g_3 follow in a similar way. Rewrite

$$g_1 = f_1 \mathbf{1}_{\mathbb{R}^n \setminus sh\mathcal{Q}} + \sum_L (f_1)_{\le\lambda} \mathbf{1}_L + \sum_L \frac{1}{|L|} \int_L (f_1)_{>\lambda} \mathbf{1}_L := I + II + III.$$

From the definition in (2.6) we know

$$\|f_1 \mathbf{1}_{\mathbb{R}^n \setminus sh\mathcal{Q}}\|_{\mathcal{Y}_3} = 0 \lesssim \lambda^{1-\frac{q}{3}}.$$

Moreover, it is easy to see that

$$II = f_1 \mathbf{1}_{f_1 \le \lambda \cap sh\mathcal{Q}} = f_1 \mathbf{1}_S,$$

where

$$S = f_{1 \le \lambda} \cap sh\mathcal{Q}.$$

Combining (2.6) and using the Hölder's inequality, we have

$$\|f_1 \mathbf{1}_S\|_{\mathcal{Y}_3} = \sup_L \inf_{x \in \widehat{L}} M_2 f_1 \mathbf{1}_S = \sup_L \inf_{x \in \widehat{L}} \sup_{x \in Q} \Big(\frac{1}{|Q|} \int_{S \cap Q} |f_1|^3 \Big)^{\frac{1}{3}} \le \lambda^{1-\frac{q}{3}} \|f_1\|_{\dot{\mathcal{X}}_q} \le \lambda^{1-\frac{q}{3}}.$$

Now we are in the position to consider III. It is easy to see that

$$III \le \sum_L \frac{1}{|\widehat{L}|} \int_{\widehat{L}} (f_1)_{>\lambda} \mathbf{1}_L \le \sum_L \inf_{x \in \widehat{L}} M_q f_1 \mathbf{1}_L \le \sum_L \mathbf{1}_L.$$

Therefore, by the fact

$$\| \sum_L \mathbf{1}_L \|_{\mathcal{Y}_3} \le 1 \le \lambda^{1-\frac{q}{3}},$$

it follows that

$$\|g_1\|_{\mathcal{Y}_3} \lesssim \lambda^{1-\frac{q}{3}}.$$

This finishes the proof of Lemma 2.2. □

3. Analysis of the kernel

In Section 2, we discussed the generalized kernel K. Here we specialize to rough kernels. For fixed Ω in $L^r(\mathbb{S}^{2n-1})$ we consider the kernel

$$K(u,v) = \frac{\Omega\big((u,v)/|(u,v)|\big)}{|(u,v)|^{2n}}. \tag{3.1}$$

We introduce the relevant notation. Define $\|[K]\|_r$ and $w_{j,r}[K]$ as follows:

$$\|[K]\|_r := \sup_{s \in \mathbb{Z}} 2^{\frac{2sn}{r'}} \big(\|K_s(u,v)\|_{L^r(\mathbb{R}^{2n})} \big),$$

$$w_{j,r}[K] = \sup_{s \in \mathbb{Z}} 2^{\frac{2sn}{r'}} \sup_{h \in \mathbb{R}^n, |h| < 2^{s-j-c_m}} \big(\|K_s(u,v) - K_s(u+h, v+h)\|_{L^r(\mathbb{R}^{2n})} \big).$$

From the work in [1], we know that if the kernel satisfies $\|[K]\|_r < \infty$ and $\sum_{j=1}^{\infty} w_{j,r}[K] < \infty$, then the assumption (2.8) of Lemma 2.1 holds. However, it is difficult to verify $\|[K]\|_r < \infty$ and $\sum_{j=1}^{\infty} w_{j,r}[K] < \infty$ in the case $K(u,v) = \Omega((u,v)/|(u,v)|)|(u,v)|^{-2n}$ with $\Omega \in L^r(\mathbb{S}^{2n-1})$ for $r \neq \infty$. We overcome this difficulty by using the method of Littlewood-Paley decomposition. That is, we decompose $K = \sum\limits_{j=-\infty}^{\infty} K_j$ and then actually show that each K_j satisfies the above properties. We establish below a key lemma concerning the rough kernel $K(u,v) = \Omega((u,v)/|(u,v)|)|(u,v)|^{-2n}$.

A bilinear Calderón-Zygmund kernel L (see [20]) is a function defined away from the diagonal on $\mathbb{R}^{2n}$ that satisfies (for some bound $A > 0$)

(1) the size condition

$$|L(u,v)| \le \frac{A}{\big|(u,v)\big|^{2n}}, \qquad (u,v) \neq 0$$

(2) the smoothness condition

$$|L\big((u,v)-(u',v')\big) - L(u,v)| \le \frac{A|(u',v')|^{\epsilon}}{\big|(u,v)\big|^{2n+\epsilon}},$$

when $0 < \frac{3}{2}|(u',v')| \le |(u,v)|$, $0 < \epsilon < 1$. Such kernels give rise to bilinear Calderón-Zygmund operators that commute with translations in the following way:

$$S(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} L(x-x_1, x-x_2) f(x_1) g(x_2)\, dx_1\, dx_2.$$

Unfortunately, if Ω lies in $L^r(\mathbb{S}^{2n-1})$ with $r < \infty$, then the associated K given by (3.1) is not a bilinear Calderón-Zygmund kernel because property (2) does not hold in general, but we can decompose it as a sum of Calderón-Zygmund kernels. Given a rough bilinear kernel $K(u,v) = \Omega((u,v)/|(u,v)|)|(u,v)|^{-2n}$ as in (3.1), we decompose it as follows. We fix a smooth function α in $\mathbb{R}^+$ such that $\alpha(t) = 1$, for $t \in (0,1]$, $\alpha(t) \in (0,1)$, for $t \in (1,2)$ and $\alpha(t) = 0$, for $t \in [2,\infty)$. For $(u,v) \in \mathbb{R}^{2n}$ and $j \in \mathbb{Z}$ we introduce the functions

$$\beta(u,v) = \alpha\big(|(u,v)|\big) - \alpha\big(2|(u,v)|\big).$$
$$\beta_j(u,v) = \beta\big(2^{-j}(u,v)\big).$$

We denote Δ_j the Littlewood-Paley operator $\Delta_j f = \mathcal{F}^{-1}(\beta_j \widehat{f})$. Here and throughout this paper $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which is defined via

$$\mathcal{F}^{-1}(g)(x) = \int_{\mathbb{R}^n} g(\xi) e^{2\pi i x\cdot\xi} d\xi = \widehat{g}(-x),$$

where $\widehat{g}$ is the Fourier transform of g. Denote

$$K^i = \beta_i K \tag{3.2}$$

and

$$K^i_j = \Delta_{j-i} K^i \tag{3.3}$$

for $i,j \in \mathbb{Z}$. Then we decompose the kernel K as follows:

$$K = \sum_{j=-\infty}^{\infty} K_j, \qquad \text{with } K_j = \sum_{i=-\infty}^{\infty} K^i_j. \tag{3.4}$$

The following lemma plays a crucial role in our analysis.

Lemma 3.1. *Let $K(u,v) = \Omega((u,v)/|(u,v)|)|(u,v)|^{-2n}$ and $\Omega \in L^r(\mathbb{S}^{2n-1})$, $1 < r \le \infty$, $j \in \mathbb{Z}$. Then for any $0 < \epsilon < 1$, there is a constant $C_{n,\epsilon}$ such that the function*

$$(u,v) \mapsto K_j(u,v) = \sum_{i \in \mathbb{Z}} K^i_j(u,v)$$

is a bilinear Calderón-Zygmund kernel with bound $A \le C_{n,\epsilon} \|\Omega\|_{L^r} 2^{\max(0,j)(\epsilon+2n/r)}$.

Proof. We need to show

$$|K_j(u,v)| \le C_{n,\epsilon}\|\Omega\|_{L^r}\frac{2^{\max(0,j)(\epsilon+2n/r)}}{|(u,v)|^{2n}}, \tag{3.5}$$

$$|K_j\big((u,v)-(u',v')\big)-K_j(u,v)| \le C_{n,\epsilon}\|\Omega\|_{L^r}\frac{2^{\max(0,j)(\epsilon+2n/r)}|(u',v')|^{\epsilon}}{|(u,v)|^{2n+\epsilon}}, \tag{3.6}$$

when $0<\frac{3}{2}|(u',v')|\le|(u,v)|$.

Given $x,y\in\mathbb{R}^{2n}$ with $|x|\ge\frac{3}{2}|y|>0$, we claim that inequality (3.6) follows from

$$|K_j^i(x-y)-K_j^i(x)| \le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\min\Big(1,\frac{|y|}{2^{i-j}}\Big)\frac{2^{\max(0,j)2n/r}}{2^{-i\epsilon}2^{\min(j,0)\epsilon}|x|^{2n+\epsilon}} \tag{3.7}$$

for some $\epsilon\in(0,1)$ and all $i,j\in\mathbb{Z}$.

To show this claim, let us assume for the time being that inequality (3.7) is true. Pick an integer N^* such that $(\log_2|y|)+j\le N^*<(\log_2|y|)+j+1$. We need to consider two cases $j\ge 0$ and $j<0$.

The Case for $j\ge 0$. If $j\ge0$, then i satisfies $2^{i-j}\le|y|$, which means $i\le N^*$. Therefore, we have

$$\begin{aligned}\sum_{i\le N^*}|K_j^i(x-y)-K_j^i(x)| &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i\le N^*}\frac{2^{j2n/r}}{2^{-i\epsilon}|x|^{2n+\epsilon}}\\ &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{j(\epsilon+2n/r)}|y|^{\epsilon}}{|x|^{2n+\epsilon}}.\end{aligned}$$

If $j\ge0$, then for i satisfies $2^{i-j}>|y|$, which implies that $i>N^*$, it holds that

$$\begin{aligned}\sum_{i>N^*}|K_j^i(x-y)-K_j^i(x)| &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i>N^*}\frac{|y|}{2^{i-j}}\frac{2^{j2n/r}}{2^{-i\epsilon}|x|^{2n+\epsilon}}\\ &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{j(\epsilon+2n/r)}|y|^{\epsilon}}{|x|^{2n+\epsilon}}.\end{aligned}$$

The case for $j<0$. If $j<0$, then for $i\le N^*$, it holds that

$$\begin{aligned}\sum_{i\le N^*}|K_j^i(x-y)-K_j^i(x)| &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i\le N^*}\frac{1}{2^{-i\epsilon}2^{j\epsilon}|x|^{2n+\epsilon}}\\ &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{|y|^{\epsilon}}{|x|^{2n+\epsilon}}.\end{aligned}$$

If $j<0$, then for $i>N^*$, we obtain

$$\begin{aligned}\sum_{i>N^*}|K_j^i(x-y)-K_j^i(x)| &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i>N^*}\frac{|y|}{2^{i-j}}\frac{1}{2^{-i\epsilon}2^{j\epsilon}|x|^{2n+\epsilon}}\\ &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{|y|^{\epsilon}}{|x|^{2n+\epsilon}}.\end{aligned}$$

Combining these estimates yields

$$|K_j(x-y)-K_j(x)| \le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{\max(0,j)(\epsilon+2n/r)}|y|^\epsilon}{|x|^{2n+\epsilon}}$$

and this finishes the proof of the claim.

Therefore, to prove inequality (3.6), it is sufficient to prove (3.7). For $i\in\mathbb{Z}$, and $x\in\mathbb{R}^{2n}$, it is easy to see that

$$|K^i(x)| \le \frac{\Omega\big(x/|x|\big)}{|x|^{2n}}\mathbf{1}_{\frac{1}{2}\le\frac{|x|}{2^i}\le 2}(x).$$

Hence,

$$\|K^i\|_{L^r(\mathbb{R}^{2n})} \le \frac{1}{2^{2in}}\Big(\int_{2^{i-1}}^{2^{i+1}}\int_{\mathbb{S}^{2n-1}}|\Omega(\theta)|^r a^{2n-1}d\theta da\Big)^{\frac{1}{r}} \approx 2^{-2in/r'}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}.$$

Let $\Psi(x)=(1+|x|)^{-2n-1}$ be defined on $\mathbb{R}^{2n}$. Note that

$$|\mathcal{F}^{-1}(\beta_{i-j})(x)| \le C_\beta 2^{-2(i-j)n}(1+2^{-(i-j)}|x|)^{-2n-1} = C_\beta\Psi_{i-j}(x),$$

then, using Hölder's inequality, it yields that $K_j^i = K^i*\mathcal{F}^{-1}(\beta_{i-j})$ enjoys the following property

$$|K_j^i(x-ty)| \lesssim \|K^i\|_{L^r}\Big(\int_{2^{i-1}\le|z|\le2^{i+1}}|\Psi_{i-j}(x-ty-z)|^{r'}dz\Big)^{\frac{1}{r'}}, \tag{3.8}$$

for $x,y\in\mathbb{R}^{2n}$ and $t\in[0,1]$.

Let $z=2^iz'$, for $x,y\in\mathbb{R}^{2n}$, it follows that

$$\begin{aligned}
&\Big(\int_{2^{i-1}\le|z|\le2^{i+1}}\Big(\frac{2^{-2(i-j)n}}{(1+2^{-(i-j)}|x-ty-z|)^{2n+1}}\Big)^{r'}dz\Big)^{\frac{1}{r'}}\\
&\lesssim\Big(\int_{\frac{1}{2}\le|z'|\le2}\frac{1}{(1+2^j|\frac{x-ty}{2^i}-z'|)^{(2n+1)r'}}dz'\Big)^{\frac{1}{r'}}2^{-2(i-j)n}2^{\frac{2in}{r'}}\\
&:= N_i^j(x,y,t).
\end{aligned}$$

If $j\le 0$, then

$$N_i^j(x,y,t) \lesssim \frac{C_{n,\epsilon}}{\big(1+2^j\max\{|\frac{x-ty}{2^i}|,1\}\big)^{2n+\epsilon}}2^{-2(i-j)n}2^{\frac{2in}{r'}} \lesssim C_{n,\epsilon}\frac{2^{2in/r'}2^{i\epsilon}}{2^{j\epsilon}|x|^{2n+\epsilon}}.$$

If $j>0$, we claim that

$$N_i^j(x,y,t) \lesssim C_{n,\epsilon}\frac{2^{2jn/r}2^{2in/r'}2^{i\epsilon}}{|x|^{2n+\epsilon}}.$$

Indeed, for $\frac{1}{4}\le|\frac{x-ty}{2^i}|\le 4$, it holds that

$$N_i^j(x,y,t) \lesssim 2^{-\frac{2in}{r}}2^{\frac{2jn}{r}} \le C_{n,\epsilon}\frac{2^{-\frac{2in}{r}}2^{\frac{2jn}{r}}}{\big(1+|\frac{x-ty}{2^i}|\big)^{2n+\epsilon}} \lesssim C_{n,\epsilon}\frac{2^{2jn/r}2^{2in/r'}2^{i\epsilon}}{|x|^{2n+\epsilon}}.$$

As for the case $|\frac{x-ty}{2^i}|>4$ or $|\frac{x-ty}{2^i}|<\frac{1}{4}$, it follows that

$$N_i^j(x,y,t)\lesssim \frac{C_{n,\epsilon}}{\big(1+2^j\max\{|\frac{x-ty}{2^i}|,1\}\big)^{2n+\epsilon}}2^{-2(i-j)n}2^{\frac{2in}{r'}}\lesssim C_{n,\epsilon}\frac{2^{2in/r'}2^{i\epsilon}}{|x|^{2n+\epsilon}}.$$

Combining the above estimates, we deduce that

$$|K_j^i(x-ty)|\lesssim C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{\max(0,j)2n/r}}{2^{-i\epsilon}2^{\min(j,0)\epsilon}|x|^{2n+\epsilon}}.$$

This inequality further implies that

$$|K_j^i(x-y)-K_j^i(x)|\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{\max(0,j)2n/r}}{2^{-i\epsilon}2^{\min(j,0)\epsilon}|x|^{2n+\epsilon}} \tag{3.9}$$

On the other hand

$$\begin{aligned}\big|K_j^i(x-y)-K_j^i(x)\big| &= \Big|\int_{\mathbb{R}^{2n}}K^i(z)\int_0^1 2^{-2(i-j)n}(\nabla\mathcal{F}^{-1}\beta)(\frac{x-ty-z}{2^{i-j}})\frac{y}{2^{i-j}}\,dt\,dz\Big|\\ &\le C_{n,\epsilon}\frac{|y|}{2^{i-j}}\int_0^1\int_{\mathbb{R}^{2n}}\big|K^i(z)\big|\frac{2^{2(j-i)n}}{(1+2^{j-i}|x-ty-z|)^{2n+1}}\,dt\,dz\\ &\le C_{n,\epsilon}\frac{|y|}{2^{i-j}}\int_0^1(|K^i|*\Psi_{i-j})(x-ty)\,dt\\ &\le C_{n,\epsilon}\frac{|y|}{2^{i-j}}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{\max(0,j)2n/r}}{2^{-i\epsilon}2^{\min(j,0)\epsilon}|x|^{2n+\epsilon}}.\end{aligned}$$

This estimate, together with inequality 3.9, yields the inequality 3.7 and hence inequality 3.6 holds.

For the size condition (3.5), we may let $t=0$ in (3.8). Thus

$$\begin{aligned}\sum_{i\in\mathbb{Z}}|K_j^i(x)| &\le C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i\in\mathbb{Z}}\Big(\int_{\frac12\le|z'|\le2}\frac{1}{(1+2^j|\frac{x}{2^i}-z'|)^{(2n+\epsilon)r'}}dz'\Big)^{\frac{1}{r'}}2^{-2(i-j)n}\\ &\lesssim C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i<\widetilde{N}^*}2^{-2(i-j)n}\Big(\int_{\frac12\le|z'|\le2}\frac{1}{\big(1+2^j|\frac{x}{2^i}-z'|\big)^{(2n+\epsilon)r'}}dz'\Big)^{\frac{1}{r'}}\\ &\quad + C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sum_{i>\widetilde{N}^*}2^{-2(i-j)n}\\ &\lesssim C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{1}{|x|^{2n}}+C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{\max(0,j)2n/r}}{2^{\min(j,0)\epsilon}|x|^{2n+\epsilon}}\sum_{i<\widetilde{N}^*}2^{i\epsilon}\\ &\lesssim C_{n,\epsilon}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\frac{2^{\max(0,j)(2n/r+\epsilon)}}{|x|^{2n}},\end{aligned}$$

where $\widetilde{N}^*$ is the number such that $2^{\widetilde{N}^*}\approx 2^{\min(j,j/r')}|x|$.

Therefore, we know that K_j is a bilinear Calderón-Zygmund kernel with bound $C_{n,\epsilon}\|\Omega\|_{L^r}2^{\max(0,j)(\epsilon+2n/r)}$. The proof of this lemma is finished. $\square$

4. THE PROOF OF THEOREM 1.1

We begin by stating a known result.

Proposition 4.1 ([17]). *Let $1 \le p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let Ω be in $L^r(\mathbb{S}^{2n-1})$ with $1 < r \le \infty$ and let $\delta \in (0, 1/r')$. Let T_j be the bilinear Calderón-Zygmund operator with kernel K_j. Them, for $j \le 0$, the operator T_j maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with norm $C\|\Omega\|_{L^r(\mathbb{S}^{2n-1})} 2^{-|j|(1-\delta)}$.*

The following lemma will be crucial in dealing with the adjoints of T_Ω. The ingredients of its proof are standard but the precise statement below may not have appeared in the literature.

Lemma 4.2. *Let $1 \le r < 4$, $\delta > 0$, and let b be a smooth function on $\mathbb{R}^{2n}$ which satisfies:*

(a) $\|b\|_{L^r(\mathbb{R}^{2n})} \le C_*$,
(b) $|b(\xi, \eta)| \le C_* \min(|(\xi, \eta)|, |(\xi, \eta)|^{-\delta})$,
(c) $|\partial^\alpha b(\xi, \eta)| \le C_\alpha C_* \min(1, |(\xi, \eta)|^{-\delta})$.

Let β be a smooth function supported in an annulus in $\mathbb{R}^{2n}$ and let $\beta_j(y, z) = \beta\big(2^{-j}(y, z)\big)$ for $j \in \mathbb{Z}$. Then the multiplier

$$b_j(\xi, \eta) = \sum_{i \in \mathbb{Z}} \beta_{j-i}(\xi, \eta) b(2^i(\xi, \eta))$$

satisfies

$$\|T_{b_j}\|_{L^2 \times L^2 \to L^1} \lesssim j\, C_*\, 2^{-\delta j(1-\frac{r}{4})}.$$

Proof. Denote $b_{j,0} = \beta_j(\xi, \eta) b(\xi, \eta)$ and write $b_j = b_j^1 + b_j^2$, where b_j^1 is the diagonal part of b_j according to the wavelet decomposition in [19, Section 4] and b_j^2 is the off-diagonal part. (In this reference b is denoted by m, b_j by m_j and $b_{j,0}$ by $m_{j,0}$.)

Let

$$C_0 = \max_{|\alpha| \le \lfloor \frac{2n}{4-r'} \rfloor + 1} \|\partial^\alpha b_{j,0}\|_{L^\infty} \lesssim C_* 2^{-\delta j},$$

where C_* depends on the frequency support of the function β and n. By [19, Section 4], we obtain

$$\|T_{b_j^1}\|_{L^2 \times L^2 \to L^1} \lesssim j C_0^{1-\frac{r}{4}} \|b_{j,0}\|_{L^r}^{\frac{r}{4}} \lesssim j C_0^{1-\frac{r}{4}} \|b\|_{L^r}^{\frac{r}{4}} \lesssim j (C_* 2^{-\delta j})^{1-\frac{r}{4}} \|b\|_{L^r}^{\frac{r}{4}} \lesssim j C_* (2^{-\delta j})^{1-\frac{r}{4}}.$$

A similar estimate (without j) holds for the off-diagonal part $T_{b_j^2}$ by the same procedure as in [17, Section 5]. It follows that

$$\|T_{b_j^2}\|_{L^2 \times L^2 \to L^1} \lesssim 2^{-\delta j} \|b_{j,0}\|_{L^r(\mathbb{R}^{2n})} \lesssim C_* 2^{-\delta j}.$$

Combining the estimates for b_j^1 and b_j^2, we obtain

$$\|T_{b_j}\|_{L^2 \times L^2 \to L^1} \lesssim j C_* 2^{-\delta j(1-\frac{r}{4})}.$$

□

We also need the following lemma.

Lemma 4.3. *Let $2 \le p_1, p_2 \le \infty$, $1 \le p \le 2$, $1/p = 1/p_1 + 1/p_2$, $\Omega \in L^r(\mathbb{S}^{2n-1})$. For $j > 0$ we have that*

$$\|T_j\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \begin{cases} Cj2^{-j\delta(1-\frac{r'}{4})}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}, & \frac{4}{3} < r \le 2, \delta < \frac{1}{r'}; \\ Cj2^{-j\delta\frac{1}{2}}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}, & r > 2, \delta < 1/2. \end{cases}$$

Proof. The techniques of the proof are borrowed from [19]. Introduce the notation:

$$m = \widehat{K^0}, \quad m_j = \widehat{K_j}, \quad m_{j,0} = \widehat{K^0}\beta_j,$$

where K^0, β_j, and K_j are the same as in (3.2), (3.3), and (3.4) are associated with the fixed Ω in $L^r(\mathbb{S}^{2n-1})$.

We first fix r satisfying $4/3 < r \le 2$. As $r \le 2$, the Hausdorff-Young inequality yields that

$$\|m\|_{L^{r'}} \le \|K^0\|_{L^r} \lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})}.$$

Also, it is not too hard to verify that conditions (b) and (c) in Lemma 4.2 hold (see [19, Lemma 6.4]) with $C_* = \|\Omega\|_{L^r(\mathbb{S}^{2n-1})}$ and $\delta < 1/r'$. Applying Lemma 4.2 we obtain

$$\|T_{m_j}\|_{L^2\times L^2\to L^1} \lesssim j2^{-\delta j(1-\frac{r'}{4})}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}.$$

Now let

$$(m_j)^{*1}(\xi_1,\xi_2) = m_j(-(\xi_1+\xi_2),\xi_2), \quad (m_j)^{*2} = m_j(\xi_1,-(\xi_1+\xi_2))$$

be the two adjoint multipliers associated with m_j. Then we have

$$(m_j)^{*1} = \sum_i (\beta_{j-i}\circ A^t)\,(\widehat{\beta_i K}\circ A^t) = \sum_i (\beta_{j-i}\circ A^t)\,\widehat{\beta K}(A^t 2^i(\cdot))$$

where $A = \begin{pmatrix} -I_n & -I_n \\ 0 & I_n \end{pmatrix}$, and I_n is the $n\times n$ identity matrix.

We now notice that the function $b(\xi,\eta) = \widehat{\beta K}(A^t(\xi,\eta))$ satisfies the hypotheses of Lemma 4.2 as $A^t(\xi,\eta)$ has the same size as (ξ,η). (Here (ξ,η) is thought of as a column vector.) The same argument works for the other adjoint of m_j with the matrix $\begin{pmatrix} I_n & 0 \\ -I_n & -I_n \end{pmatrix}$ in place of A. It follows that

$$\|T_{(m_j)^{*1}}\|_{L^2\times L^2\to L^1} + \|T_{(m_j)^{*2}}\|_{L^2\times L^2\to L^1} \lesssim j2^{-j\delta(1-\frac{r'}{4})}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}.$$

By duality, we have

$$\|T_{m_j}\|_{L^\infty\times L^2\to L^2} + \|T_{m_j}\|_{L^2\times L^\infty\to L^2} \lesssim j2^{-j\delta(1-\frac{r'}{4})}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}.$$

For $4/3 < r \le 2$, interpolating between the above two estimates implies that

$$\|T_{m_j}\|_{L^{p_1}\times L^{p_2}\to L^p} \lesssim j2^{-j\delta(1-\frac{r'}{4})}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}, \quad \delta < \frac{1}{r'},$$

where $2 \le p_1, p_2 \le \infty$, $1 \le p \le 2$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Now for $r > 2$, thanks to the embedding $L^r(\mathbb{S}^{2n-1}) \subseteq L^2(\mathbb{S}^{2n-1})$, we have

$$\|T_{m_j}\|_{L^{p_1}\times L^{p_2}\to L^p} \lesssim j2^{-j\delta(1-\frac{2}{4})}\|\Omega\|_{L^2(\mathbb{S}^{2n-1})} \lesssim j2^{-j\delta\frac{1}{2}}\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}, \quad \delta < \frac{1}{2},$$

where $2 \leq p_1, p_2 \leq \infty$, $1 \leq p \leq 2$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

This completes the proof of this lemma. $\square$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. By Littlewood-Paley decomposition of the kernel, T_Ω can be written as

$$T_\Omega(f_1, f_2)(x) = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K_j(x-y, x-z) f_1(y) f_2(z)\, dydz := \sum_{j=-\infty}^{\infty} T_j(f_1, f_2)(x).$$

Given a stopping collection $\mathcal{Q}$ with top cube Q, let $\mathcal{Q}_j$ be defined as

$$\Lambda_{\mathcal{Q}_{j,t_1}^{t_2}}(f_1, f_2, f_3) = \frac{1}{|Q|}\Big[\langle T[K_j]_{t_1}^{t_2 \wedge s_Q}(f_1 \mathbf{1}_Q, f_2), f_3\rangle - \sum_{\substack{L \in \mathcal{Q} \\ L \subset Q}} \langle T[K_j]_{t_1}^{t_2 \wedge s_L}(f_1 \mathbf{1}_L, f_2), f_3\rangle\Big].$$

For the sake of simplicity, let's denote $\Lambda_{\mathcal{Q}_j}(f_1, f_2, f_3) = \Lambda_{\mathcal{Q}_{j,t_1}^{t_2}}(f_1, f_2, f_3)$.

Our proof will be divided into two parts $\sum_{j>0} T_j$ and $\sum_{j \leq 0} T_j$. Each part should satisfy the assumption (2.8) of Lemma 2.1. We therefore consider these two parts into two steps.

Step 1. Estimate for $j > 0$.

Fix $0 < \gamma < 1$, by Lemma 3.1, T_j is a bilinear Calderón-Zygmund operator with kernel K_j, and the size and smoothness conditions constant $A_j \leq C_{n,\gamma} \|\Omega\|_{L^r} 2^{j(\gamma + 2n/r)}$.

Combining the methods in [1, Section 3], we know the kernel of T_j satisfies $\|[K_j]\|_p \lesssim 2^{j(\epsilon + 2n/r)} < \infty$ for fixed $j \in \mathbb{Z}$. This enables us to use Lemma 3.1 and Proposition 3.3 in [1] with $A_j \leq C_{n,\epsilon} \|\Omega\|_{L^r} 2^{j(\gamma + 2n/r)}$ (Then choose $\beta = 1$ and $p = 1$). Hence

$$|\Lambda_{\mathcal{Q}_j}(f_1, f_2, f_3)| \lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} 2^{j(\gamma + 2n/r)} |Q| \|f_1\|_{\dot{\mathcal{X}}_1} \|f_2\|_{\mathcal{Y}_1} \|f_3\|_{\mathcal{Y}_1}.$$

By Lemma 4.3, choosing $p_1 = p_2 = 3$, we have

$$|\Lambda_{\mathcal{Q}_j}(f_1, f_2, f_3)| \lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} j 2^{-cj} |Q| \|f_1\|_{\dot{\mathcal{X}}_3} \|f_2\|_{\mathcal{Y}_3} \|f_3\|_{\mathcal{Y}_3},$$

where $c < 1/r'(1 - r'/4)$, if $4/3 < r \leq 2$ and $c < 1/4$ if $r > 2$.

Interpolating via Lemma 2.2, it follows that for any $0 < \epsilon < 1$ there exits $q = 1 + 2\epsilon$ so that

$$\begin{aligned} |\Lambda_{\mathcal{Q}_j}(f_1, f_2, f_3)| &\lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} 2^{j(\gamma + 2n/r)(1-\epsilon)} j^\epsilon 2^{-cj\epsilon} |Q| \|f_1\|_{\dot{\mathcal{X}}_q} \|f_2\|_{\mathcal{Y}_q} \|f_3\|_{\mathcal{Y}_q} \\ &\lesssim j 2^{-j\gamma\epsilon} 2^{j(\gamma + 2n/r)} 2^{-(c + 2n/r)j\epsilon} \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} |Q| \|f_1\|_{\dot{\mathcal{X}}_q} \|f_2\|_{\mathcal{Y}_q} \|f_3\|_{\mathcal{Y}_q}. \end{aligned}$$

If we choose $\gamma < c$ and $\epsilon = \frac{2n/r + \gamma}{2n/r + c}$, then $0 < \epsilon < 1$. Therefore

$$|\Lambda_{\mathcal{Q}_j}(f_1, f_2, f_3)| \lesssim j 2^{-j\gamma\epsilon} |Q| \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} \|f_1\|_{\dot{\mathcal{X}}_q} \|f_2\|_{\mathcal{Y}_q} \|f_3\|_{\mathcal{Y}_q}.$$

Summing over $j \in \mathbb{Z}^+$, we can conclude that for $q = 1 + 2\frac{2n/r + \gamma}{2n/r + c}$

$$|\Lambda_{\mathcal{Q}}(f_1, f_2, f_3)| \lesssim |Q| \|\Omega\|_{L^r(\mathbb{S}^{2n-1})} \|f_1\|_{\dot{\mathcal{X}}_q} \|f_2\|_{\mathcal{Y}_q} \|f_3\|_{\mathcal{Y}_q}.$$

By symmetry, it also yields that

$$|\Lambda_{\mathcal{Q}}(f_1,f_2,f_3)| \lesssim |Q|\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\|f_1\|_{\mathcal{Y}_q}\|f_2\|_{\dot{\mathcal{X}}_q}\|f_3\|_{\mathcal{Y}_q},$$
$$|\Lambda_{\mathcal{Q}}(f_1,f_2,f_3)| \lesssim |Q|\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\|f_1\|_{\mathcal{Y}_q}\|f_2\|_{\mathcal{Y}_q}\|f_3\|_{\dot{\mathcal{X}}_q}.$$

Step 2. Estimate for $j \le 0$.

By Lemma 3.1, T_j is a bilinear Calderón-Zygmund kernel with constant $A_j \le \|\Omega\|_{L^r(\mathbb{S}^{2n-1})}$. Hence

$$|\Lambda_{\mathcal{Q}j}(f_1,f_2,f_3)| \lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})}|Q|\|f_1\|_{\dot{\mathcal{X}}_1}\|f_2\|_{\mathcal{Y}_1}\|f_3\|_{\mathcal{Y}_1}.$$

By Proposition 4.1 with $p_1 = p_2 = 2$, we have

$$|\Lambda_{\mathcal{Q}j}(f_1,f_2,f_3)| \lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})}2^{-c|j|}|Q|\|f_1\|_{\dot{\mathcal{X}}_2}\|f_2\|_{\mathcal{Y}_2}\|f_3\|_{\mathcal{Y}_\infty},$$

where $c = 1-\delta$, $\delta < 1/r'$. For any $q > 1$, by Lemma 4.3 and Lemma 4.4 in [1], then summing over $j \le 0$, one obtains

$$|\Lambda_{\mathcal{Q}}(f_1,f_2,f_3)| \lesssim |Q|\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\|f_1\|_{\dot{\mathcal{X}}_q}\|f_2\|_{\mathcal{Y}_q}\|f_3\|_{\mathcal{Y}_q}.$$
$$|\Lambda_{\mathcal{Q}}(f_1,f_2,f_3)| \lesssim |Q|\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\|f_1\|_{\mathcal{Y}_q}\|f_2\|_{\dot{\mathcal{X}}_q}\|f_3\|_{\mathcal{Y}_q}.$$
$$|\Lambda_{\mathcal{Q}}(f_1,f_2,f_3)| \lesssim |Q|\|\Omega\|_{L^r}\|f_1\|_{\mathcal{Y}_q}\|f_2\|_{\mathcal{Y}_q}\|f_3\|_{\dot{\mathcal{X}}_q}.$$

In conclusion, the above two steps hold for

$$p > \begin{cases} \frac{24n+3r-4}{8n+3r-4}, & \frac{4}{3} < r \le 2; \\ \frac{24n+r}{8n+r}, & r > 2. \end{cases}$$

since the norm of $\mathcal{Y}_q$ is increasing over q.

Using Theorem A, we can find r_1, r_2 in $[2,\infty]$ and α in $[1,2]$ such T_Ω maps $L^{r_1}\times L^{r_2}$ to L^α. But a smooth truncation of the kernel $K(u,v)$ also gives rise to an operator with a similar bound (see Remark 2.1), thus we have that $C_T(r_1,r_2,\alpha) < \infty$ and (2.4) is valid. Hence, T_Ω satisfies Lemma 2.1. Moreover, we can choose $c < \frac{1}{r'}(1-\frac{r'}{4})$ if $\frac{4}{3} < r \le 2$, and $c < \frac{1}{4}$ if $r > 2$, such that $p > 3 - \frac{2c}{2n/r+c}$. Then

$$|\Lambda_{\mathcal{Q}}(f_1,f_2,f_3)| \lesssim \|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\sup_{\mathcal{S}}\mathsf{PSF}_{\mathcal{S},\vec{p}}(f_1,f_2,f_3),$$

this finishes the proof of Theorem 1.1, since the multiplication operators regarding the remaining truncations satisfy the required $\mathsf{PSF}_{\mathcal{S}}^{(1,1,1)}$ bound [1, Section 6.2]. □

5. Derivation of the Corollaries

Proof of Corollary 1.2. The techniques are borrowed from [10], but the weight classes are different.

Define $\sigma = v_{\vec{w}}^{-\frac{q'}{q}}$ and choose $p_i > \max\{\frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\}$, with $p_i < q_i$, $i = 1,2$ and $p_3' > q$. By Theorem 1.1 and duality, for any sparse collection $\mathcal{S}$, it is enough to show that

$$\mathsf{PSF}_{\mathcal{S}}^{(p_1,p_2,p_3)}(f_1,f_2,f_3) \lesssim \prod_{i=1}^{2}\|f_i\|_{L^{q_i}(v_i)}\|f_3\|_{L^{q'}(\sigma)} \tag{5.1}$$

with bounds independent of $\mathcal{S}$.

Let

$$w_1 = v_1^{\frac{p_1}{p_1-q_1}}, \quad w_2 = v_2^{\frac{p_2}{p_2-q_2}}, \quad w_3 = \sigma^{\frac{p_3}{p_3-q'}}$$

and $f_i = g_i w_i^{\frac{1}{p_i}}$, $i = 1,2,3$. Then we have

$$\|f_i\|_{L^{q_i}(v_i)} = \|g_i\|_{L^{q_i}(w_i)}, \qquad i = 1,2,$$

and

$$\|f_3\|_{L^{q'}(\sigma)} = \|g_3\|_{L^{q'}(w_3)}.$$

Let $q_3 = q'$. It follows that

$$\begin{aligned}
&\mathsf{PSF}_{\mathcal{S}}^{(p_1,p_2,p_3)}(f_1,f_2,f_3)\\
&= \mathsf{PSF}_{\mathcal{S}}^{(p_1,p_2,p_3)}\big(g_1 w_1^{\frac{1}{p_1}}, g_2 w_2^{\frac{1}{p_2}}, g_3 w_3^{\frac{1}{p_3}}\big)\\
&= \sum_{Q\in\mathcal{S}} \Big(\prod_{j=1}^{3} w_j(E_Q)^{\frac{1}{q_j}} \Big(\frac{\langle g_j^{p_j} w_j\rangle_Q}{\langle w_j\rangle_Q}\Big)^{\frac{1}{p_j}}\Big) \times \Big(\prod_{j=1}^{3} \big(\langle w_j\rangle_Q\big)^{\frac{1}{p_j}-\frac{1}{q_j}}\Big) \times \Big(|Q| \prod_{j=1}^{3} \Big(\frac{\langle w_j\rangle_Q}{w_j(E_Q)}\Big)^{\frac{1}{q_j}}\Big).
\end{aligned}$$

By a simple calculation, we have

$$\prod_{j=1}^{2} \langle w_j\rangle_Q^{\frac{1}{p_j}-\frac{1}{q_j}} \langle w_3\rangle_Q^{\frac{1}{p_3}-\frac{1}{q'}} = \prod_{j=1}^{2} \langle w_j\rangle_Q^{\frac{1}{p_j}-\frac{1}{q_j}} \langle v_{\vec{w}}^{\frac{p_3'}{p_3'-q}}\rangle_Q^{\frac{1}{q}-\frac{1}{p_3'}} = [\vec{v}]_{A_{\vec{q},\vec{p}}}.$$

We now deal with the second product using the technique in [27]. Let

$$x_1 = \frac{p_1-q_1}{p_1q_1}, \quad x_2 = \frac{p_2-q_2}{p_2q_2}, \quad x_3 = \frac{p_3-q'}{p_3q'},$$

then

$$w_1^{-\frac{x_1}{2}} w_2^{-\frac{x_2}{2}} w_3^{-\frac{x_3}{2}} = 1.$$

Hölder's inequality and the fact that

$$-\frac{x_1}{2} - \frac{x_2}{2} - \frac{x_3}{2} + \frac{1}{2p_1'} + \frac{1}{2p_2'} + \frac{1}{2p_3'} = 1$$

imply that

$$\prod_{i=1}^{3} \big(w_i(E_Q)\big)^{-\frac{x_i}{2}} E_Q^{\frac{1}{2p_i'}} \geqslant \int_{E_Q} \prod_{i=1}^{3} w_i^{-\frac{x_i}{2}} = |E_Q|.$$

The sparseness of $\mathcal{S}$ yields that

$$\prod_{i=1}^{3} \Big(\frac{w_i(E_Q)}{|Q|}\Big)^{-\frac{x_i}{2}} \geq \eta^{-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}}.$$

Therefore

$$\prod_{i=1}^{3} \Big(\frac{\langle w_i\rangle_Q}{\frac{1}{|Q|} w_i(E_Q)}\Big)^{-\frac{x_i}{2}} \leq \eta^{\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}} \prod_{i=1}^{3} \langle w_i\rangle_Q^{-\frac{x_i}{2}}.$$

By Definition1.3, we have

$$\prod_{i=1}^{3}\Big(\frac{\langle w_i\rangle_Q}{\frac{1}{|Q|}w_i(E_Q)}\Big)^{\frac{1}{q_i}} \le \big(\eta^{x_1+x_2+x_3}\prod_{i=1}^{3}\langle w_i\rangle_Q{}^{-x_i}\big)^{\max(-\frac{1}{x_iq_i})}$$
$$\le \big(\eta^{x_1+x_2+x_3}[\vec{v}]_{A_{\vec{q},\vec{p}}}\big)^{\max(-\frac{1}{x_iq_i})}.$$

Finally note that, by [10], the first product depends on the $L^{q_j}(w_j)$-boundedness of M_{p_j,w_j}, where

$$M_{p_j,w_j}f(x) = \sup_{Q\ni x}\Big(\frac{1}{|w(Q)|}\int_Q |f|^{p_j}w_j\Big)^{\frac{1}{p_j}}.$$

This concludes the proof of (5.1) □

Proof of Corollary 1.3. For $2 < p < \infty$, let $\sigma = w^{\frac{-2}{2-p}}$, $\rho = \frac{p}{p-2}$ and choose $q_i > \max\{\frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\}$ such that $q_i < \rho$, and $q_i < p$. By Theorem 1.1 and duality, it is enough to prove that for any sparse collection $\mathcal{S}$, we have

$$\mathsf{PSF}_{\mathcal{S}}^{(q_1,q_2,q_3)}(f_1,f_2,f_3) \lesssim \prod_{i=1}^{2}\|f_i\|_{L^p(w)}\|f_3\|_{L^\rho(\sigma)}$$

with bounds independent of $\mathcal{S}$. The proof of this fact is omitted as it follows from the same steps as in Section 5.1 in [1]. □

Next, we provide another corollary which is related to Corollary 1.7 in [10].

Corollary 5.1. *Suppose $\Omega \in L^r(\mathbb{S}^{2n-1})$ with vanishing integral and $r > 4/3$. For $p_1, p_2 > \max\{\frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $1 < p < \max\{\frac{24n+3r-4}{16n}, \frac{24n+r}{16n}\}$. Then for weights $w_1^2 \in A_{p_1}$, $w_2^2 \in A_{p_2}$, $w = w_1^{\frac{p}{p_1}}w_2^{\frac{p}{p_2}}$, there exists a constant $C = C_{w,p_1,p_2,n,r}$ such that*

$$\|T_\Omega(f_1,f_2)\|_{L^p(w)} \le C\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}.$$

We end this section with another corollary concerning the commutator of a rough T_Ω with a pair of BMO functions $\vec{b} = (b_1, b_2)$. For a pair $\vec{\alpha} = (\alpha_1, \alpha_2)$ of nonnegative integers, we define this commutator (acting on a pair of nice functions f_j) as follows:

$$\big[T_\Omega, \vec{b}\,\big]_{\vec{\alpha}}(f_1,f_2)(x) = \text{p.v.}\int_{\mathbb{R}^{2n}}\frac{\Omega((y_1,y_2)')}{|(y_1,y_2)|^{2n}}f_1(x-y_1)f_2(x-y_2)\prod_{i=1}^{2}(b_i(x)-b_i(y_i))^{\alpha_i}dy_1dy_2$$

As a consequence of Proposition 5.1 in [31] and of Corollary 1.2,

Corollary 5.2. *Let $\Omega \in L^r(\mathbb{S}^{2n-1})$ with $r > 4/3$ and $\int_{\mathbb{S}^{2n-1}}\Omega\, d\sigma = 0$. Let $\vec{q} = (q_1, q_2)$, $\vec{p} = (p_1, p_2, p_3)$ with $\vec{p} \prec \vec{q}$ and $p_i > \max\{\frac{24n+3r-4}{8n+3r-4}, \frac{24n+r}{8n+r}\}$, $i = 1, 2, 3$. Let*

$$\mu_{\vec{v}} = \prod_{k=1}^{2} v_k^{q/q_k}$$

and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $1 < q < \max\{\frac{24n+3r-4}{16n}, \frac{24n+r}{16n}\}$ *and let* $q_3 = q'$. *Then there is a constant* $C = C_{\vec{p},\vec{q},r,n,\vec{\alpha}}$ *such that*

$$\left\| \left[T_\Omega, \vec{b}\,\right]_{\vec{\alpha}}(f_1, f_2) \right\|_{L^q(\mu_{\vec{v}})} \leq C\|\Omega\|_{L^r(\mathbb{S}^{2n-1})}[\vec{v}]_{A_{\vec{q},\vec{p}}}^{\max_{1\leq i\leq 3}\{\frac{p_i}{q_i-p_i}\}}\|f_1\|_{L^{q_1}(v_1)}\|f_2\|_{L^{q_2}(v_2)}\prod_{i=1}^{2}\|b_i\|_{BMO}^{\alpha_i}.$$

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