2.1 Regions of the Complex Plane

$$A \cap B = \{z : z \in A \text{ and } z \in B\}.$$

Two sets *A* and *B* are **disjoint** if $A \cap B = \emptyset$. The **set difference** between *A* and *B* is the set

$$A \setminus B = \{z : z \in A \text{ and } z \notin B\}.$$

We say that *A* is a **subset** of *B* or that *B* contains *A* and we write $A \subset B$ or $B \supset A$ if every element of *A* is also an element of *B*: $z \in A \Rightarrow z \in B$.

Connected Sets

A basic result from calculus states that if the derivative of a differentiable function is constant on an open interval, then the function is constant vanishes on that interval. This result is not true if the domain of definition of the function is not connected. For example, consider the function

$$h(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ -1 & \text{if } 2 < t < 3, \end{cases}$$

whose domain of definition is $(0,1) \cup (2,3)$. We have h'(t) = 0 for all *t* in the union $(0,1) \cup (2,3)$, but clearly *h* is not constant. This example shows that the connectedness of the domain is required in order to have consistent behavior and be able to derive important conclusions for functions. For subsets of the plane, one way to define connectedness is as follows.

Definition 2.1.5. A **polygonal line** is a finite union of closed line segments L_j , j = 1, ..., m, such that the end of each L_j coincides with the beginning of L_{j+1} (j = 1, ..., m-1). A subset Ω of the complex plane is called **polygonally connected** if any two points in Ω can be joined by a polygonal line entirely contained in it.

We denote by $[z_0, z_1]$ the line segment

$$\{(1-t)z_0 + tz_1 : t \in [0,1]\}$$

that joins two fixed points $z_0, z_1 \in \mathbb{C}$. A polygonal line $\bigcup_{j=1}^{m} [z_{j-1}, z_j]$ can be thought of as a map *L* from the interval [0,m] to the complex plane defined as follows:

$$L(t) = (j-t)z_{j-1} + (t-j+1)z_j$$

for $t \in [j - 1, j], j = 1, ..., m$. Then

$$L(j) = z_j,$$

i.e., L(t) passes through the point z_j at "time" t = j. See Figure 2.5.



Fig. 2.5 A polygonally connected subset and a polygonal line (m = 3).