as small as we wish. Also, if we can find one value of r, say r_0, such that $B_{r_0}(z)$ is contained in S, then we can find infinitely many values of r such that $B_r(z)$ is contained in S: Just take $0 < r < r_0$. Here are some useful examples to keep in mind.

- The empty set, denoted as usual by $\emptyset$, is open. Because there are no points in $\emptyset$, the definition of open sets is vacuously satisfied.
- The set of all complex numbers $\mathbb{C}$ is open.
- An r-neighborhood, $B_r(z_0)$, is open. We just verified in Example 2.1.3(a) that every point in $B_r(z_0)$ is an interior point.
- The set of all z such that $|z - z_0| > r$ is open. This set is called a **neighborhood of** ∞.

An r-neighborhood, $B_r(z_0)$, is more commonly called an **open disk** of radius r, centered at z_0.

One can show that a set is open if and only if it contains none of its boundary points (Exercise 18). Sets that contain all of their boundary points are called **closed**. The complex plane $\mathbb{C}$ and the empty set $\emptyset$ are closed since they trivially contain their empty sets of boundary points. The disk $\{z : \ |z - z_0| \le r\}$ is closed because it contains all its boundary points consisting of the circle $|z - z_0| = r$ (Figure 2.2). We refer to such a disk as the **closed disk** of radius r, centered at z_0. The smallest closed set that contains a set A is called the **closure** of A and is denoted by $\overline{A}$. For instance the closure of the open disk $B_r(z_0)$ is the closed disk $\overline{B_r(z_0)} = \{z : \ |z - z_0| \le r\}$. The punctured open disk $B'_r(z_0)$ also has the same closure. A point z_0 is called an **accumulation point** of a set A if $B'_r(z_0) \cap A \neq \emptyset$ for every $r > 0$. For instance, every boundary point of an open disk is an accumulation point of it.

Some sets are neither open nor closed. For example, the set

$$S = \{z : \ |z - z_0| \le r; \ \operatorname{Im} z > 0\}$$

contains the boundary points on the upper semicircle, but it does not contain its boundary points that lie on the x-axis. Hence, this set is neither open nor closed. See Figure 2.4.

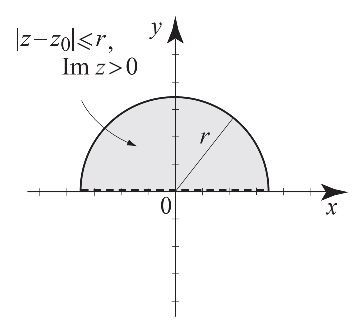

Fig. 2.4 S is neither open nor closed.

Next, we introduce some set notation for convenience. If a point z is in a set S, we say that z is an **element** of S and write $z \in S$. If z does not belong to S, we will write $z \notin S$. Let A and B be two sets of complex numbers. The **union** of A and B, denoted $A \cup B$, is the set

$$A \cup B = \{z : \ z \in A \text{ or } z \in B\}.$$

The **intersection** of A and B, denoted $A \cap B$, is the set