- If x is a positive real number, then $\operatorname{Log} x = \ln x$.
- If x is a negative real number, then $\operatorname{Log} x = \ln |x| + i\pi$.
- For all z, the identity $e^{\operatorname{Log} z} = z$ is true. But, as illustrated by Example 1.8.3(e), $\operatorname{Log}(e^z)$ is not always equal to z. In fact, $\operatorname{Log}(e^z) = z \Leftrightarrow -\pi < \operatorname{Im} z \leq \pi$.
- Many algebraic properties of $\ln x$ no longer hold for $\operatorname{Log} z$. For example, the identity $\ln(x_1 x_2) = \ln x_1 + \ln x_2$, which holds for all positive real numbers x_1 and x_2, does not hold for $\operatorname{Log} z$. Consider the following:

$$\operatorname{Log}\big((-1)(-1)\big) = \operatorname{Log}(1) = 0 \neq \operatorname{Log}(-1) + \operatorname{Log}(-1),$$

  since $\operatorname{Log}(-1) = i\pi$.

Branches of the argument and the logarithm

As we may imagine, we could have specified a different range of values of $\arg z$ in defining a logarithmic function in terms of (1.8.3). In fact, for every real number α we can specify that $\alpha < \arg z \leq \alpha + 2\pi$. This selection assigns a single value to $\arg z$, denoted by $\arg_\alpha z$, that lies in the interval $(\alpha, \alpha + 2\pi]$.

Definition 1.8.4. Let α be a fixed real number. For $z \neq 0$, we call the unique value of $\arg z$ that falls in the interval $(\alpha, \alpha + 2\pi]$ *the α-th branch of* $\arg z$ and we denote it by $\arg_\alpha z$. Precisely, we define the α-th **branch** of $\log z$ by the identity

$$\log_\alpha z = \ln |z| + i \arg_\alpha z, \qquad \text{where } \alpha < \arg_\alpha z \leq \alpha + 2\pi. \tag{1.8.6}$$

The ray through the origin along which a branch of the logarithm is discontinuous is called a **branch cut**.

When $\alpha = -\pi$, this definition leads to the principal value of the logarithm; that is, $\log_{-\pi} z = \operatorname{Log} z$.

Since two values of $\arg z$ differ by an integer multiple of 2π, it follows that, for a complex number $z \neq 0$ and real numbers α and β, there is an integer k (depending on z, α, and β), such

$$\not{n}\,\log_\alpha z = \log_\beta z + 2k\pi i\,.$$

Example 1.8.5. (Different branches of the logarithm) Evaluate
(a) $\log_0 i$ (b) $\log_{\frac{\pi}{2}} i$ (c) $\log_{\frac{\pi}{2}}(-2)$

Solution. If we know $\log z$, to find $\log_\alpha z$, it suffices to choose the value of $\log z$ with imaginary part that lies in the interval $(\alpha, \alpha + 2\pi]$. If we do not know $\log z$, we compute $\log_\alpha z$ using (1.8.6).
(a) We have $\alpha = 0$ and so the imaginary part of $\log_0 z$, $\arg_0 z$, must be in the interval $(0, 2\pi]$. From Example 1.8.1, $\log i = i\frac{\pi}{2} + 2k\pi i$; and so $\log_0 i = i\frac{\pi}{2}$. Note that $\operatorname{Log} i =$