(c) We have

$$|-2| = 2,\ \arg(-2) = \pi + 2k\pi \Rightarrow \log(-2) = \ln 2 + i(\pi + 2k\pi) = \ln 2 + (2k+1)\pi i\,.$$

More explicitly, $\log(-2)$ consists of the following complex values:

$$\ln 2 + \pi i,\ \ln 2 - \pi i,\ \ln 2 + 3\pi i,\ \ln 2 - 3\pi i,\ \ln 2 \pm 5\pi i, \ldots .$$

Note that all values of $\log(-2)$ have identical real parts and their imaginary parts differ by integer multiples of $2\pi i$. These observations are true in general. □

It is clear from Example 1.8.1 and from the definition of $\log z$ that to make $\log z$ single-valued, and hence turn it into a function, it is enough to define a single-valued version of $\arg z$. For example, we can use the principal value of the argument, $\operatorname{Arg} z$ (see Definition 1.3.2) which satisfies $-\pi < \operatorname{Arg} z \leq \pi$.

Definition 1.8.2. The **principal value** or **principal branch** of the complex logarithm is defined by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z \qquad (z \neq 0). \tag{1.8.4}$$

Thus $\operatorname{Log} z$ is the (particular) value of $\log z$ whose imaginary part is in the interval $(-\pi, \pi]$. Because $\arg z = \operatorname{Arg} z + 2k\pi$, where k is an integer, we see from (1.8.3) and (1.8.4) that all the values of $\log z$ differ from the principal value by $2k\pi i$. Thus

$$\log z = \operatorname{Log} z + 2k\pi\, i \qquad (z \neq 0). \tag{1.8.5}$$

Example 1.8.3. (Principal values of the logarithm) Compute the expressions
(a) $\operatorname{Log} i$ (b) $\operatorname{Log}(1+i)$ (c) $\operatorname{Log}(-i)$ (d) $\operatorname{Log} 5$ (e) $\operatorname{Log}(e^{6\pi i})$.

Solution. If we know $\log z$, to find $\operatorname{Log} z$, it suffices to choose the value of $\log z$ with imaginary part that lies in the interval $(-\pi, \pi]$. If we do not know $\log z$, we compute $\operatorname{Log} z$ using (1.8.4). Appealing to Example 1.8.1, we have for (a) $\operatorname{Log} i = i\,\frac{\pi}{2}$; and for (b) $\operatorname{Log}(1+i) = \operatorname{Log}(1+i) = \frac{1}{2}\ln 2 + i\,\frac{\pi}{4}$. For (c), we use (1.8.4):

$$|-i| = 1,\ \operatorname{Arg}(-i) = -\frac{\pi}{2} \Rightarrow \operatorname{Log}(-i) = \ln(1) - i\frac{\pi}{2} = -i\frac{\pi}{2}.$$

For (d), we use (1.8.4):

$$|5| = 5,\ \operatorname{Arg} 5 = 0 \Rightarrow \operatorname{Log} 5 = \ln 5.$$

For (e), we use (1.8.4) and note that $e^{6\pi i}$ is a unimodular number. In fact $e^{6\pi i} = 1$. So

$$|e^{6\pi i}| = 1,\ \operatorname{Arg}(e^{6\pi i}) = 0 \Rightarrow \operatorname{Log}(e^{6\pi i}) = \ln 1 = 0. \qquad \square$$

A few observations are in order to highlight some similarities and differences between the natural logarithm and the complex logarithm.