Looking for an extension of the exponential function e^x ($x \in \mathbb{R}$) to the complex plane, we wonder whether we are able to substitute x with a complex number z in (1.6.1). Note that the series in (1.6.1) converges absolutely by an ~~easy~~ application of the ratio test (Example 1.5.24). Consequently, for a complex z we have

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = 1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots = e^{|z|} < \infty.$$

Thus the series of complex numbers $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely and therefore it converges by Theorem 1.5.20.

Definition 1.6.1. We define the **complex exponential function** $\exp(z)$ or e^z as the convergent series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \text{for all } z \in \mathbb{C}. \tag{1.6.2}$$

We discuss some fundamental properties of the complex exponential function.

Theorem 1.6.2. *Let z and w be arbitrary complex numbers. We have that*

$$e^{z+w} = e^z e^w \tag{1.6.3}$$

Moreover, e^z is never zero and

$$e^{-z} = \frac{1}{e^z} \tag{1.6.4}$$

$$e^{z-w} = \frac{e^z}{e^w} \tag{1.6.5}$$

Proof. We have $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$, where both series converge absolutely. Applying Theorem 1.5.28 we obtain

$$\Big(\sum_{k=0}^{\infty} \frac{z^k}{k!}\Big)\Big(\sum_{m=0}^{\infty} \frac{w^m}{m!}\Big) = \sum_{n=0}^{\infty} c_n \tag{1.6.6}$$

and c_n is defined in (1.5.7) by

$$c_n = \sum_{j=0}^{n} \frac{z^j}{j!} \frac{w^{n-j}}{(n-j)!} = \frac{1}{n!} \overbrace{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j}}^{(z+w)^n} = \frac{(z+w)^n}{n!}, \tag{1.6.7}$$

where the last equality is a consequence of the binomial identity (Exercise 65 in Section 1.3). Now the left hand side of (1.6.6) is $e^z e^w$, but, in view of (1.6.7), the right hand side is e^{z+w}, hence (1.6.3) holds.

Now (1.6.4) is a consequence of (1.6.3) and of the fact that $e^0 = 1$ since

$$1 = e^{z-z} = e^z e^{-z}.$$