In Exercises 21–32*, determine whether the series is convergent or divergent.*

21. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1+3i}{4}\right)^n$ **22.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n+4^n}{(1+3i)^n}$ **23.** $\displaystyle\sum_{n=0}^{\infty}\frac{3i^n}{4+in^2}$

24. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3+10i}{4+5in}\right)^n$ **25.** $\displaystyle\sum_{n=1}^{\infty}\frac{(1+2in)^n}{n^n}$ **26.** $\displaystyle\sum_{n=1}^{\infty}\frac{\left(\frac{2+i}{2-i}\right)^n}{n^2}$

27. $\displaystyle\sum_{n=1}^{\infty}\operatorname{Re}\left[\left(\cos(\tfrac{1}{n^3})+i\sin(\tfrac{1}{n^3})\right)^n\right]$ **28.** $\displaystyle\sum_{n=1}^{\infty}\operatorname{Im}\left[\left(\cos(\tfrac{1}{n^3})+i\sin(\tfrac{1}{n^3})\right)^n\right]$ **29.** $\displaystyle\sum_{n=1}^{\infty}\frac{e^n-ie^{-n}}{e^{n^2}}$

30. $\displaystyle\sum_{n=1}^{\infty}\frac{(3+10i)n^n}{n!}$ **31.** $\displaystyle\sum_{n=0}^{\infty}\frac{(2+3i)^n}{n!}$ **32.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3+i^n}$

In Exercises 33–40*, use the geometric series to determine the largest* ~~*region*~~ *open set in which the series converges and find the value of the infinite sum.*

33. $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{2^n}$ **34.** $\displaystyle\sum_{n=1}^{\infty}(1+z)^n$ **35.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{(3+i)z}{4-i}\right)^n$

36. $\displaystyle\sum_{n=0}^{\infty}\frac{(2+i)^n}{z^n}$ **37.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2-10z)^n}$ **38.** $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+1}}{(2+i-z)^n}$

39. $\displaystyle\sum_{n=0}^{\infty}\left\{\left(\frac{2}{z}\right)^n+\left(\frac{z}{3}\right)^n\right\}$ **40.** $\displaystyle\sum_{n=0}^{\infty}\left\{\frac{1}{(1-z)^n}-z^n\right\}$

41. The nth partial sum of a series is $s_n=\frac{i}{n}$. Does the series converge or diverge? If it does converge, what is its limit?

42. Show that if $\sum_{n=0}^{\infty}a_n$ is absolutely convergent, then $|\sum_{n=0}^{\infty}a_n|\le\sum_{n=0}^{\infty}|a_n|$.

43. Let $t>0$ and x be real numbers. Find the sum $\sum_{n=0}^{\infty}e^{-nt}\cos nx$. [Hint: Proceed as in Example 1.5.16.]

44. (a) The nth term of a series is $1-\frac{1}{n}$. Does the series converge?
(b) The nth partial sum of a series is $1+\frac{1}{n}$. Does the series converge?

45. The terms of a series are defined recursively by

$$a_1=2+i,\qquad a_{n+1}=\frac{(7+3i)n}{1+2in^2}a_n.$$

Does the series $\sum_{n=1}^{\infty}a_n$ converge or diverge?

46. The terms of a series are defined recursively by

$$a_1=i,\qquad a_{n+1}=\frac{\cos(\frac{1}{n})+i\sin(\frac{1}{n})}{\sqrt{n}}a_n.$$

Does the series $\sum_{n=1}^{\infty}a_n$ converge or diverge?

1.6 The Complex Exponential

For a real number x, we recall how to express e^x in a series as follows:

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\qquad(-\infty<x<\infty).\tag{1.6.1}$$