Here is a simple application of the comparison test, which illustrates the passage from complex to real series in establishing the convergence of complex series.

Example 1.5.22. (Comparison test) The series $\sum_{n=0}^{\infty} \frac{2\cos(n\theta)+2i\sin(n\theta)}{n^2+3}$ is convergent by comparison to the convergent series $\sum_{n=1}^{\infty} \frac{2}{n^2}$, because

$$\left|\frac{2\cos(n\theta)+2i\sin(n\theta)}{n^2+3}\right| \leq \frac{2\,|\cos(n\theta)+i\sin(n\theta)|}{n^2} = \frac{2}{n^2}. \qquad \square$$

Theorem 1.5.23. (Ratio Test) *Let a_n be nonzero complex numbers and suppose that*

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \tag{1.5.5}$$

exists or is infinite. Then the complex series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\rho < 1$ and diverges if $\rho > 1$. If $\rho = 1$ the test is inconclusive.

Example 1.5.24. (Ratio test~~and the exponential series~~) The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely for all z. The series is obviously convergent if $z = 0$. For $z \neq 0$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}\,n!}{z^n\,(n+1)!}\right| = \lim_{n\to\infty}\frac{|z|}{n+1} = 0.$$

Since $\rho < 1$, the series is absolutely convergent by the ratio test, hence it is convergent. $\square$

Theorem 1.5.25. (Root Test) *Let a_n be complex numbers and suppose that*

$$\rho = \lim_{n\to\infty}|a_n|^{1/n} \tag{1.5.6}$$

either exists or is infinite. Then the complex series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\rho < 1$ and diverges if $\rho > 1$. If $\rho = 1$ the test is inconclusive.

In general, the ratio test is easier to apply than the root test. But there are situations that call naturally for the root test. Here is an example.

Example 1.5.26. Test the series $\sum_{n=0}^{\infty} \frac{z^n}{(n+1)^n}$ for convergence.

Solution. The presence of the exponent n in the terms suggests using the root test. We have

$$\rho = \lim_{n\to\infty}\left|\frac{z^n}{(n+1)^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{|z|}{n+1} = 0.$$

Since $\rho < 1$, the series is absolutely convergent for all z. $\square$