$$s_m + t_m = \sum_{n=0}^{\infty} a_n = s \quad \Rightarrow \quad t_m = s - s_m.$$

Let $m \to \infty$ and use $s_m - s \to 0$ to obtain that $t_m \to 0$, as desired. ■

Definition 1.5.19. A complex series $\sum_{n=0}^{\infty} a_n$ is said to be **absolutely convergent** if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

A well-known consequence of the completeness property of real numbers is that every bounded monotonic sequence (increasing or decreasing) converges. Since the partial sums of a series with nonnegative terms are increasing, we conclude that if these partial sums are bounded, then the series is convergent. Thus, if for a complex series we have $\sum_{n=1}^{N} |a_n| \le M$ for all N, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Recall that, for series with real terms, absolute convergence implies convergence. The same is true for complex series.

Theorem 1.5.20. *Absolutely convergent series are convergent, i.e., for $a_n \in \mathbb{C}$*

$$\sum_{n=0}^{\infty} |a_n| < \infty \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n \text{ converges}.$$

Proof. Let $s_n = a_0 + a_1 + \cdots + a_n$ and $v_n = |a_0| + |a_1| + \cdots + |a_n|$. By Theorem 1.5.11, it is enough to show that the sequence of partial sums $\{s_n\}_{n=0}^{\infty}$ is Cauchy. For $n > m \ge 0$, using the triangle inequality, we have

$$|s_n - s_m| = \left| \sum_{j=m+1}^{n} a_j \right| \le \sum_{j=m+1}^{n} |a_j| = v_n - v_m.$$

Since $\sum_{n=0}^{\infty} |a_n|$ converges, the sequence $\{v_n\}_{n=0}^{\infty}$ converges and hence it is Cauchy. Thus, given $\varepsilon > 0$ we can find N so that, $v_n - v_m < \varepsilon$ for $n > m \ge N$, implying that $|s_n - s_m| < \varepsilon$ for $n > m \ge N$. Hence $\{s_n\}_{n=0}^{\infty}$ is a Cauchy sequence. ■

For a complex series $\sum_{n=0}^{\infty} a_n$, consider the series $\sum_{n=0}^{\infty} |a_n|$ whose terms are real and nonnegative. If we can establish the convergence of the series $\sum_{n=0}^{\infty} |a_n|$ using one of the tests of convergence for series with nonnegative terms, then using Theorem 1.5.20, we can infer that the series $\sum_{n=0}^{\infty} a_n$ is convergent. Thus, all known tests of convergence for series with nonnegative terms can be used to test the (absolute) convergence of complex series. We list a few such convergence theorems.

Theorem 1.5.21. *Suppose that a_n are complex numbers, b_n are real numbers, $|a_n| \le b_n$ for all $n \ge n_0$, and $\sum_{n=0}^{\infty} b_n$ is convergent. Then $\sum_{n=0}^{\infty} a_n$ is ~~absolutely~~ convergent.*

Proof. By the comparison test for real series, we have that $\sum_{n=0}^{\infty} |a_n|$ is convergent. By Theorem 1.5.20, it follows that $\sum_{n=0}^{\infty} a_n$ is convergent. ■