We can use complex series to sum real series.

Example 1.5.16. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$ converges for all real θ and find the sum.

Solution. We recognize $\cos n\theta$ as the real part of $(\cos\theta + i\sin\theta)^n$, and so the given series is the real part of the geometric series

$$\sum_{n=0}^{\infty} z^n \quad \text{where} \quad z = \frac{1}{2}(\cos\theta + i\sin\theta).$$

From Example 1.5.13, since $|z| = 1/2 < 1$, we have

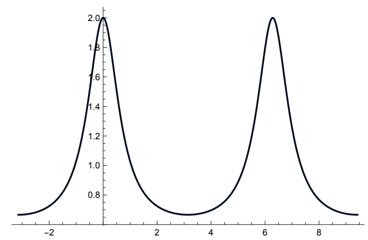

Fig. 1.36 Graph of the series $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$ over $[-\pi, 3\pi]$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1-\bar{z}}{(1-z)(1-\bar{z})} = \frac{1 - \frac{1}{2}\cos\theta + \frac{i}{2}\sin\theta}{(1-\frac{1}{2}\cos\theta)^2 + (\frac{1}{2}\sin\theta)^2} = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}.$$

Taking real parts and using Theorem 1.5.15(iii), we obtain

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = \mathrm{Re}\left(\frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}\right) = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}.$$

The series is plotted in Figure 1.36 as a function of θ. □

Theorem 1.5.17. (The nth Term Test for Divergence) *If $\sum_{n=0}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=0}^{\infty} a_n$ diverges.*

Proof. Let $s_n = \sum_{m=0}^{n} a_m$. If $s_n \to s$, then also $s_{n-1} \to s$, and so $s_n - s_{n-1} \to s - s = 0$. But $s_n - s_{n-1} = a_n$, and so $a_n \to 0$. ■

Applying the nth term test, we see right away that the geometric series $\sum_{n=0}^{\infty} z^n$ is divergent if $|z| = 1$ or $|z| > 1$.

For $m \geq 1$, the expression $t_m = \sum_{n=m+1}^{\infty} a_n$ is called a **tail of the series** $\sum_{n=0}^{\infty} a_n$. For fixed m, the tail t_m is itself a series, which differs from the original series by finitely many terms. So it is obvious that a series converges if and only if all its tails converge. As $m \to \infty$, we are dropping more and more terms from the tail series; as a result, we have the following useful fact.

Proposition 1.5.18. *If $\sum_{n=0}^{\infty} a_n$ is convergent, then $\lim_{m\to\infty} \sum_{n=m+1}^{\infty} a_n = 0$. Hence if a series converges, then its tail tends to 0.*

Proof. Let $s = \sum_{n=0}^{\infty} a_n$, $t_m = \sum_{n=m+1}^{\infty} a_n$, and $s_m = \sum_{n=1}^{m} a_n$. Since s_m is a partial sum of $\sum_{n=0}^{\infty} a_n$, we have $s_m \to s$ as $m \to \infty$. For each m, we have