So in order to establish the convergence or divergence of a series, we must study the behavior of the sequence of partial sums.

Example 1.5.13. Show that if |z| < 1 the **geometric series** $\sum_{n=0}^{\infty} z^n$ converges and

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Show that the series diverges for all other values of z.

Solution. Consider the partial sum (a typical case is shown in Figure 1.34):

$$s_n = 1 + z + z^2 + \dots + z^n$$

We multiply by z and add 1 to obtain

$$1 + zs_n = 1 + z + z^2 + z^3 + \dots + z^{n+1}$$

which is equal to $s_n + z^{n+1}$. If $z \neq 1$, we solve for s_n to find



Fig. 1.34 Terms in a convergent geometric series (|z| < 1). To get a partial sum s_n , add the vectors $1, z, ..., z^n$.

$$s_n = \frac{(1+z+z^2+\dots+z^n)(1-z)}{1-z} = \frac{1-z^{n+1}}{1-z}$$

From Example 1.5.9, the sequence $\{z^{n+1}\}_{n=0}^{\infty}$ converges to 0 if |z| < 1 and diverges if |z| > 1 or |z| = 1 and $z \neq 1$. This implies that $\{s_n\}_{n=0}^{\infty}$ converges to $\frac{1}{1-z}$ if |z| < 1 and diverges for all other values of z, which is what we wanted to show. \Box

Geometric series may appear in disguise. Basically, whenever you see a series of the form $\sum_{n=0}^{\infty} w^n$ you should be able to use the geometric series to sum it. However, you have to be careful with the region of convergence.

Example 1.5.14. (Geometric series in disguise) Determine the largest open set in which the series

$$\sum_{n=0}^{\infty} \frac{1}{(4+2z)^n}$$

is convergent and find its sum.

Solution. The series is a geometric series of the form $\sum_{n=0}^{\infty} w^n$ where $w = \frac{1}{4+2^2}$.