The following theorem is also analogous to one from calculus. Its proof is omitted.

Theorem 1.5.6. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of complex numbers.*
(i) Suppose that $\lim_{n\to\infty} a_n = 0$ and $|b_n| \le |a_n|$ for all $n \ge n_0$. Then $\lim_{n\to\infty} b_n = 0$.
(ii) If $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}_{n=1}^\infty$ is a bounded sequence then $\lim_{n\to\infty} a_n b_n = 0$.

The proof of the next theorem is also left to the reader.

Theorem 1.5.7. *If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are convergent sequences of complex numbers and α and β are complex numbers, then*

$$\begin{aligned}
&\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n;\\
&\lim_{n\to\infty}(a_n\, b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n;\\
&\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \textit{if } \lim_{n\to\infty} b_n \neq 0;\\
&\lim_{n\to\infty}\overline{a_n} = \overline{\lim_{n\to\infty} a_n};\\
&\lim_{n\to\infty}|a_n| = \big|\lim_{n\to\infty} a_n\big|.
\end{aligned}$$

Theorem 1.5.8. *Suppose that $\{z_n\}_{n=1}^\infty$ is a sequence of complex numbers and write $z_n = x_n + i\,y_n$, where $x_n = \operatorname{Re} z_n$ and $y_n = \operatorname{Im} z_n$. Then for x, y real numbers we have*

$$\lim_{n\to\infty} z_n = x + i\,y \quad \Leftrightarrow \quad \lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y.$$

Proof. Suppose that $z_n \to x + i\,y$. Then by Theorem 1.5.7 we have that $\overline{z_n} \to \overline{x+i\,y}$. Using again Theorem 1.5.7 we obtain $z_n + \overline{z_n} \to x + i\,y + \overline{x+i\,y} = 2x$ and $z_n - \overline{z_n} \to x + i\,y - \overline{(x+i\,y)} = 2iy$. Thus $2x_n \to 2x$ and $2iy_n \to 2iy$ which implies that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Conversely, if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then by Theorem 1.5.7 we have $iy_n \to iy$ and adding yields $x_n + iy_n \to x + i\,y$. ■

Next we show how to use the preceding results along with our knowledge of real-valued sequences to compute limits of complex-valued sequences.

Example 1.5.9. (A useful limit) Show that

$$\lim_{n\to\infty} z^n = \begin{cases} 0 & \text{if } |z| < 1,\\ 1 & \text{if } z = 1.\end{cases}$$

Moreover, show that the limit does not exist for all other values of z; that is, if $|z| > 1$, or $|z| = 1$ and $z \neq 1$, then $\lim_{n\to\infty} z^n$ does not exist.

Solution. Recall that for a real number $r \ge 0$, we have