$$u(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(s)}{(x-s)^2+y^2}\,ds \quad (z = x+iy),$$

which is Poisson's formula for the upper half-plane. □

We give one more example of a Green's function.

Example 7.5.6. (Green's function for a semi-infinite vertical strip) Compute Green's function for a semi-infinite vertical strip.

We can map the strip Ω in Figure 7.104 conformally onto the upper half-plane using the mapping $w = \sin z$. Composing the function (7.5.12) with this, we obtain a one-to-one analytic mapping of Ω onto the unit disk, taking z in Ω onto the origin. Thus the Green's function for Ω is

$$G(z, \zeta) = \ln\left|\frac{\sin z - \sin\zeta}{\overline{\sin z} - \sin\zeta}\right|. \quad \square$$

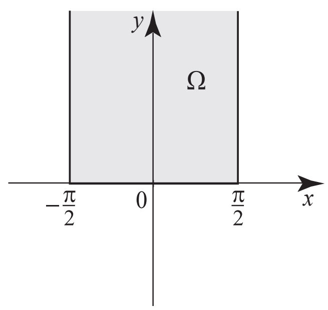

Fig. 7.104 A semi-infinite vertical strip.

Next we prove ~~next~~ some interesting properties of Green's functions.

Theorem 7.5.7. (Properties of Green's Functions) *Suppose that Ω is a simply connected region with boundary Γ, and let ϕ, $\Phi(z, \zeta)$, and $G(z, \zeta)$ be as in Theorem 7.5.2. Then the Green's function G has the following properties:*
(i) $G(z, \zeta) \le 0$ for all z and ζ in Ω;
(ii) $G(z, \zeta) = 0$ for all z in Ω and ζ on Γ;
*(iii) $G(z, \zeta) = G(\zeta, z)$ for all z and ζ in Ω (**symmetric property**);*
(iv) for each z in Ω, there is a function $\zeta \mapsto u_1(z, \zeta)$ such that $u_1(z, \zeta)$ is harmonic for all ζ in Ω, $u_1(z, \zeta) = -\ln|z-\zeta|$ for all ζ on the boundary Γ, and $G(z, \zeta) = u_1(z, \zeta) + \ln|z-\zeta|$ for all $\zeta \ne z$ in Ω.

Properties (i) and (ii) could be verified on the graphs of the Green's functions in Figures 7.102 and 7.103. Before we prove the theorem, we illustrate the properties in Figure 7.105 for a typical case where Ω is the upper half-plane and Green's function is anchored at $z = 1+i$.