$$\left|\frac{z-\zeta}{\overline{z}-\zeta}\right| = \left|\frac{\overline{z-\zeta}}{\overline{z}-\overline{\zeta}}\right| = \frac{|z-\zeta|}{\left|\overline{z-\zeta}\right|} = 1.$$

Thus τ maps the real line onto the unit circle and since it takes z onto the origin, it follows that τ maps the upper half-plane onto the unit disk, and thus $\tau(\zeta) = \Phi(z, \zeta)$ for the upper half-plane.

Example 7.5.5. (Green's function and Poisson's formula in the upper half-plane) (a) Show that the Green's function for the upper half-plane is

$$G(z, \zeta) = \frac{1}{2}\ln\frac{(x-s)^2+(y-t)^2}{(x-s)^2+(y+t)^2}, \quad \text{for } z = x+iy,\ \zeta = s+it\ (y, t > 0). \tag{7.5.13}$$

Fix $z = 1+i$ in the upper half-plane, and plot the function $\zeta \mapsto G(1+i, \zeta)$, for ζ in the upper half-plane. This is the Green's function for the upper half-plane anchored at a specific point $z = 1+i$ in the upper half-plane.
(b) Derive the Poisson integral formula for the upper half-plane.

Solution. (a) According to (7.5.10), the Green's function for the upper half-plane is $\ln|\Phi(z, \zeta)|$, where $\Phi(z, \zeta)$ is in (7.5.12). Thus,

$$\begin{aligned} G(z, \zeta) &= \ln\left|\frac{z-\zeta}{\overline{z}-\zeta}\right| \\ &= \frac{1}{2}\ln\frac{|z-\zeta|^2}{|\overline{z}-\zeta|^2} \\ &= \frac{1}{2}\ln\frac{(x-s)^2+(y-t)^2}{(x-s)^2+(-y-t)^2}, \end{aligned}$$

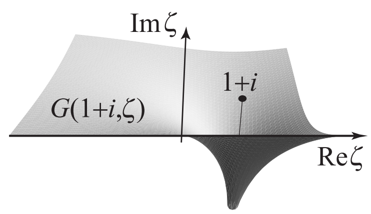

Fig. 7.103 Green's function $G(1+i, \zeta)$ for the upper half-plane anchored at $z = 1+i$. Note that $G(1+i, \zeta) = 0$ for all ζ on the boundary and $G(1+i, \zeta)$ has a singularity at $\zeta = 1+i$.

which is equivalent to (7.5.13).The function $G(1+i, \zeta)$ is plotted in Figure 7.103.
(b) To derive Poisson's integral formula in the upper half-plane we compute the normal derivative in (7.5.9). If Γ is the real s-axis, then the normal derivative is clearly the derivative in the negative direction along the imaginary t-axis. Thus,

$$\frac{\partial}{\partial n}G(z, \zeta) = -\frac{1}{2}\frac{\partial}{\partial t}\ln\frac{(x-s)^2+(y-t)^2}{(x-s)^2+(y+t)^2}.$$

A straightforward calculation of the derivative, then setting $t = 0$, yields

$$\frac{\partial}{\partial n}G(z, \zeta) = \frac{2y}{(x-s)^2+y^2}.$$

Plugging into (7.5.9) yields