Using $\Phi(z,\zeta)$ in place of $\phi(\zeta)$ in (7.5.4), we are able to reproduce the value of u at any point z in Ω.

Theorem 7.5.2. (Green's Functions) *Suppose that Ω is a simply connected region with boundary Γ, and ϕ is a one-to-one analytic function on Ω and its boundary onto the unit disk and its boundary. Let u be a function that is harmonic on Ω and piecewise continuous on Γ. For z and ζ in Ω, let $\Phi(z,\zeta)$ be as in (7.5.8). Then, for any z in Ω, we have*

$$u(z) = \frac{1}{2\pi}\int_{\Gamma} u(\zeta)\frac{\partial}{\partial n}\ln|\Phi(z,\zeta)|\,ds, \tag{7.5.9}$$

where $ds = |d\zeta|$ is the element of arc length on Γ.

Definition 7.5.3. The function

$$G(z,\zeta) = \ln|\Phi(z,\zeta)| = \ln\left|\frac{\phi(\zeta)-\phi(z)}{1-\overline{\phi(z)}\,\phi(\zeta)}\right|, \quad z,\zeta \text{ in } \Omega, \tag{7.5.10}$$

is called the **Green's function** for the region Ω. Formula (7.5.9) is a **generalized Poisson integral formula** for the simply connected region Ω.

Green's functions play a fundamental role in the solution of important partial differential equations (Laplace's equation and Poisson's equation).

Like the Poisson formulas on the disk and in the upper half-plane, formula (7.5.9) can be used to solve a general Dirichlet problem in a simply connected region Ω, where the boundary data is piecewise continuous. Of course, this solution depends on the explicit formula for the conformal mapping of Ω onto the unit disk. Once this mapping is determined, Green's functions can be used to solve the Dirichlet problem. We illustrate these ideas with several examples and show how we can recapture the Poisson formulas from Green's functions.

We often write the Green's function $G(z,\zeta)$ in terms of the real and imaginary parts of $z = x+iy$ and $\zeta = s+it$. We also write the Green's function using polar coordinates of z and ζ, where $z = re^{i\theta}$ and $\zeta = \rho e^{i\eta}$

Example 7.5.4. (Green's function and Poisson formula for the disk) (a) Show that the Green's function for the unit disk in polar coordinates is

$$G(z,\zeta) = \ln\left|\frac{\rho e^{i\eta} - re^{i\theta}}{1 - r\rho e^{i(\eta-\theta)}}\right|, \quad \text{for } z = re^{i\theta} \text{ and } \zeta = \rho e^{i\eta}. \tag{7.5.11}$$

As a specific illustration, fix $z = \frac{2}{5}$ in the unit disk, and plot the function $\zeta \mapsto G(\frac{2}{5},\zeta)$, for ζ in the unit disk. This is the Green's function for the unit disk anchored at a specific point $z = \frac{2}{5}$ in the unit disk.
(b) Derive the Poisson integral formula for the unit disk.