when $n = 5$. If P is bounded, then the point w_n is taken to be the initial and terminal point of the closed path P. If P is unbounded, we take $w_n = \infty$ and think of P as a polygon with $n-1$ vertices $w_1, w_2, \dots, w_{n-1}$ (Figure 7.75). It will be convenient to measure the exterior angle at a vertex, and so we let θ_j denote the angle that we make as we turn the corner of the polygon at w_j. We choose $0 < |\theta_j| < \pi$ ($j = 1, \dots, n$); a positive value corresponds to a left turn and a negative value corresponds to a right turn. In Figure 7.74, θ_2 is negative while all other θ_j are positive.

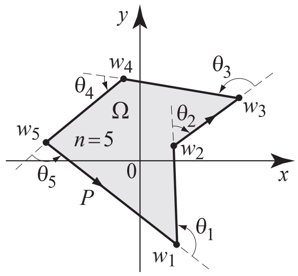

Fig. 7.74 Positively oriented polygonal boundary with corner angles measured from the outside.

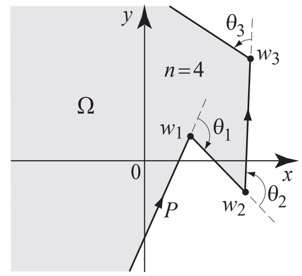

Fig. 7.75 Unbounded polygonal region with n sides ($n = 4$) and $n-1$ vertices.

Theorem 7.4.1. (Schwarz-Christoffel Transformation) *Let Ω be a region bounded by a polygonal path P with vertices at w_j (counted consecutively) and corresponding exterior angles θ_j. Then there is a one-to-one conformal mapping $f(z)$ of the upper half-plane onto Ω, such that*

$$f'(z) = A\,(z-x_1)^{-\frac{\theta_1}{\pi}}\,(z-x_2)^{-\frac{\theta_2}{\pi}} \cdots (z-x_{n-1})^{-\frac{\theta_{n-1}}{\pi}}\,, \tag{7.4.1}$$

where A is a constant, the x_j's are real and satisfy $x_1 < x_2 < \cdots < x_{n-1}$, $f(x_j) = w_j$, $\lim_{z\to\infty} f(z) = w_n$, and the complex powers are defined by their principal branches.

The points x_j ($j = 1, \dots, n-1$) on the x-axis are the pre-images of the vertices of the polygonal path P in the w-plane. Two of the x_j's may be chosen arbitrarily, so long as they are arranged in ascending order. We can express the fact that $\lim_{z\to\infty} f(z) = w_n$ by writing $f(\infty) = w_n$. In the case of an unbounded polygon P, we have $f(\infty) = \infty$.

Definition 7.4.2. The mapping f whose derivative is the function in (7.4.1) is called a **Schwarz-Christoffel transformation**, after the German mathematicians Karl Hermann Amandus Schwarz (1843–1921) and Elwin Bruno Christoffel (1829–1900). Since f is an antiderivative of the function in (7.4.1), we can write

$$f(z) = A \int (z-x_1)^{-\frac{\theta_1}{\pi}}\,(z-x_2)^{-\frac{\theta_2}{\pi}} \cdots (z-x_{n-1})^{-\frac{\theta_{n-1}}{\pi}}\,dz + B. \tag{7.4.2}$$