(e) Derive the Fourier series expansion of the triangular wave: for all θ,

$$f(\theta)=\frac{\pi}{2}+\sum_{n\text{ odd}}\frac{4}{\pi n^2}\cos(n\theta)=\frac{\pi}{2}+\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\big((2k+1)\theta\big). \tag{6.4.20}$$

(f) Illustrate the convergence of the Fourier series to $f(\theta)$ by plotting several partial sums.

8. Project Problem: Solve the Dirichlet problem on the unit disk, where the boundary values are $f(\theta)=\theta$, $0<\theta<2\pi$. In your solution, follow parts (a)-(d) of the previous exercise.

9. (a) Plot the graph over the interval $-2\pi\le\theta\le 3\pi$ of the 2π-periodic sawtooth function

$$f(\theta)=\begin{cases}\frac{1}{2}(\pi-\theta) & \text{if } 0<\theta\le 2\pi,\\ \cancel{[6pt]}f(\theta+2\pi) & \text{otherwise.}\end{cases}$$

(b) Derive the Fourier series

$$f(\theta)=\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n}.$$

10. Let the 2π-periodic function f be defined on the interval $[-\pi,\pi)$ by $f(\theta)=|\theta|$ if $-\pi\le\theta<\pi$. Derive the Fourier series $f(\theta)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\big((2k+1)\theta\big)$.

11. Let the 2π-periodic function f be defined on the interval $[-\pi,\pi)$ by

$$f(\theta)=\begin{cases}1 & \text{if}\quad 0<\theta<\pi/2,\\ -1 & \text{if}\quad -\pi/2<\theta<0,\\ 0 & \text{if}\quad \pi/2<|\theta|<\pi.\end{cases}$$

Prove that the Fourier series of this function is

$$f(\theta)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin(n\theta).$$

12. Show that the Fourier series of the 2π-periodic function $|\sin\theta|$, $-\pi\le\theta\le\pi$, is

$$\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k)^2-1}\cos(2k\theta).$$

13. Show that the Fourier series of the 2π-periodic function $|\cos\theta|$, $-\pi\le\theta\le\pi$, is

$$\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)^2-1}\cos(2k\theta).$$

14. Reflecting and translating a Fourier series. Suppose that f is 2π-periodic and let $g(\theta)=f(-\theta)$ and $h(\theta)=f(\theta-\alpha)$, where α is a fixed real number. To avoid confusion we use $a(\phi,n)$ and $b(\phi,n)$ instead of a_n and b_n to denote the Fourier coefficients of a function ϕ.
(a) Show that $a(f,0)=a(g,0)$, $a(f,n)=a(g,n)$, and $b(f,n)=-b(g,n)$ for all $n\ge 1$.
(b) Show that $a(f,0)=a(h,0)$ and that for $n\ge 1$ we have

$$\begin{aligned}a(h,n)&=a(f,n)\cos(n\alpha)-b(f,n)\sin(n\alpha)\\ b(h,n)&=a(f,n)\sin(n\alpha)+b(f,n)\cos(n\alpha).\end{aligned}$$