where a and b are real constants to be determined so as to satisfy the boundary conditions. From the first condition we obtain

$$u(x,0) = 100 \Rightarrow a\,\operatorname{Arg} x + b = 100 \Rightarrow b = 100,$$

as $\operatorname{Arg} x = 0$ for $x > 0$. From the second condition

$$\begin{aligned} u(0,y) = 50 &\Rightarrow a\,\operatorname{Arg}(iy) + b = 50 \\ &\Rightarrow a\frac{\pi}{2} + 100 = 50 \\ &\Rightarrow a = -\frac{100}{\pi}, \end{aligned}$$

since $\operatorname{Arg}(iy) = \frac{\pi}{2}$ for $y > 0$, and $b = 100$. Thus the steady-state temperature inside the plate is

$$u(x,y) = -\frac{100}{\pi}\operatorname{Arg}(z) + 100.$$

Now for $z = x + iy$ with $x > 0$, we have

$$\operatorname{Arg} z = \tan^{-1}\left(\frac{y}{x}\right),$$

and so another way of expressing the solution is

$$u(x,y) = -\frac{100}{\pi}\tan^{-1}\left(\frac{y}{x}\right) + 100.$$

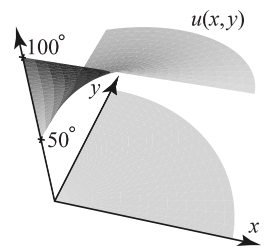

Fig. 6.11 A three-dimensional picture representing the temperature distribution of the plate. Note the boundary values on the graph.

The graph of u is shown in Figure 6.11. Note the temperature on the boundary; it matches the boundary conditions. □

In contrast to $\operatorname{Arg} z$ we can find harmonic functions which are independent of the argument and depend only on $r = |z|$. An example of such a function is $u(z) = a\ln|z| + b$, where a and b are real constants. By Example 6.1.3(d), this function is harmonic in $\mathbb{C}\setminus\{0\}$. It is a good candidate for a solution of Dirichlet problems in which the boundary data is constant on circles. See Exercises 9–12 for illustrations.

Example 6.2.2. (Dirichlet problem in an infinite strip) Solve the Dirichlet problem shown in Figure 6.12.