By the mean value property of analytic functions (3.9.4), we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})\,dt + \frac{i}{2\pi}\int_0^{2\pi} v(z+re^{it})\,dt.$$

Now take real parts on both sides to obtain (6.1.11). ■

We now prove the maximum-minimum modulus principle for harmonic functions, which resembles that for analytic functions. ~~Note the role of ... in the proof.~~

Theorem 6.1.20. (Maximum and Minimum Modulus Principle) *Suppose that u is a harmonic function on a region Ω. If u attains a maximum or a minimum in Ω, then u is constant in Ω.*

Proof. By considering $-u$, we need only prove the statement for maxima. We first prove the result under the assumption that Ω is simply connected. Applying Theorem 6.1.16 we find an analytic function $f = u + iv$ on Ω. Consider the function

$$g = e^f = e^u e^{iv}.$$

Then g is analytic in Ω and $|g| = e^u$. Since the real exponential function is strictly increasing, a maximum of e^u corresponds to a maximum of u. By Theorem 3.9.6, if $|g|$ attains a maximum or a minimum in Ω, then g is constant, implying that u is constant in Ω.

We now deal with an arbitrary region Ω. Suppose that u attains a maximum M at a point in Ω. Let

$$\begin{aligned} \Omega_0 &= \{z \in \Omega :\ u(z) < M\} \\ \Omega_1 &= \{z \in \Omega :\ u(z) = M\}. \end{aligned}$$

We have $\Omega = \Omega_0 \cup \Omega_1$, Ω_0 is open, and Ω_1 is nonempty by assumption. It is enough to show that Ω_1 is open. By connectedness this will imply that $\Omega = \Omega_1$.

Suppose that z_0 is in Ω_1 and let $B_r(z_0)$ be an open disk in Ω centered at z_0 (Figure 6.9). Since $B_r(z_0)$ is simply connected and the restriction of u to $B_r(z_0)$ is a harmonic function that attains its maximum at z_0 inside $B_r(z_0)$, it follows from the previous case that u is constant in $B_r(z_0)$. Thus $u(z) = M$ for all z in $B_r(z_0)$, implying that $B_r(z_0)$ is contained in Ω_1. Hence Ω_1 is open. ■

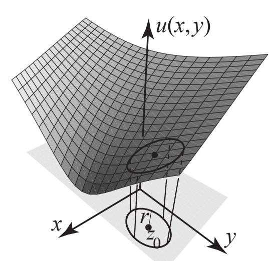

Fig. 6.9 Local existence of the harmonic conjugate.

Note that in Theorem 6.1.20 the minimum principle holds without the further assumption that $u \neq 0$, which was required for the minimum principle for analytic