Proof. Consider the analytic conjugate gradient $\phi = u_x - i u_y$. The integral of ϕ is independent of path in Ω, in view of Corollary 3.6.9. Define

$$f(z) = \int_{\gamma(z_0,z)} \phi(\zeta)\, d\zeta.$$

Then f is analytic (see (3.3.5) and what follows) and $f' = \phi$. Write $\zeta = x + iy$, $d\zeta = dx + i\,dy$. Then

$$f(z) = \int_{\gamma(z_0,z)} u_x\, dx + u_y\, dy + i \overbrace{\int_{\gamma(z_0,z)} -u_y\, dx + u_x\, dy}^{v(z)}.$$

We claim that

$$\int_{\gamma(z_0,z)} u_x\, dx + u_y\, dy = u(z) - u(z_0).$$

From this it will follow that $v(z) = \int_{\gamma(z_0,z)} -u_y\, dx + u_x\, dy$ is a harmonic conjugate of $u(z)$, since $(u - u(z_0)) + iv = f$ is analytic as the additive constant $u(z_0)$ does not affect analyticity. To prove the claim, parametrize the path from z_0 to z by $\zeta(t) = x(t) + iy(t)$, $a \le t \le b$. Then

$$u_x\, dx + u_y\, dy = \left(u_x \frac{dx}{dt} + u_y \frac{dy}{dt} \right) dt = \frac{d}{dt} u(\zeta(t))\, dt,$$

by the chain rule in two dimensions. Hence

$$\int_{\gamma(z_0,z)} u_x\, dx + u_y\, dy = \int_a^b \frac{d}{dt} u(\zeta(t))\, dt = u(\zeta(t)) \Big|_a^b = u(z) - u(z_0),$$

as claimed. ∎

Suppose that u is harmonic on a region Ω and let z_0 be in Ω. Since Ω is open, we can find an open disk $B_R(z_0)$ in Ω. Since $B_R(z_0)$ is simply connected, u has a harmonic conjugate in $B_R(z_0)$. This means that Theorem 6.1.16 holds *locally* in Ω (Figure 6.4). This is a useful fact that we record in the following corollary.

Corollary 6.1.17. *A harmonic function defined on a region has a harmonic conjugate locally.*

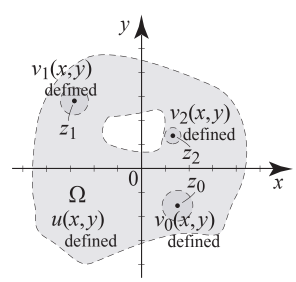

Fig. 6.4 Local existence of the harmonic conjugate.

The function u and its conjugate v have a very interesting geometric relationship based on the notion of orthogonal curves.