Adding these equations and using that $u_{xx}+u_{yy}=0$, $v_{xx}+v_{yy}=0$, and that $u_xv_x+u_yv_y=0$ (which is a consequence of the Cauchy-Riemann identity (2.5.7) in Theorem 2.5.1), we deduce that

$$\begin{aligned}(w\circ f)_{xx}+(w\circ f)_{yy} &= w_{uu}(u_x)^2+w_{uu}(u_y)^2+w_{vv}(v_x)^2+w_{vv}(v_y)^2\\ &= w_{uu}(u_x)^2+w_{uu}(v_x)^2+w_{vv}(v_x)^2+w_{vv}(u_x)^2\\ &= \big(w_{uu}+w_{vv}\big)\big((u_x)^2+(v_x)^2\big)\end{aligned}$$

and so (6.1.3) follows. If w is harmonic, then $\Delta w=0$ and it follows from (6.1.3) that $\Delta(w\circ f)=0$, and thus $w\circ f$ is harmonic if w is harmonic. ■

Harmonic Conjugates

Definition 6.1.11. Suppose that u and v are harmonic functions that satisfy the Cauchy-Riemann equations on some open set Ω, in other words, the function $f=u+iv$ is analytic in Ω. Then v is called the **harmonic conjugate** of u.

Can we always find a harmonic conjugate of a harmonic function u? As it turns out, the answer depends on the function u and its domain of definition. For example, the function $\ln|z|$ is harmonic in $\Omega=\mathbb{C}\setminus\{0\}$ (Example 6.1.3(d)); but $\ln|z|$ has no harmonic conjugate in that region (Exercise 34). It does, however, have a harmonic conjugate in $\mathbb{C}\setminus(-\infty,0]$, namely $\operatorname{Arg} z$. Our next example shows one way of using the Cauchy-Riemann equations to find the harmonic conjugate in a region such as the entire complex plane, a disk, or a rectangle.

Example 6.1.12. (Finding harmonic conjugates) Show that $u(x,y)=x^2-y^2+x$ is harmonic in the entire plane and find a harmonic conjugate for it.

Solution. That u is harmonic follows from $u_{xx}=2$ and $u_{yy}=-2$. To find a harmonic conjugate v, we use the Cauchy-Riemann equations as follows. We want $u+iv$ to be analytic. Hence v must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \tag{6.1.4}$$

Since $\frac{\partial u}{\partial x}=2x+1$, the first equation implies that

$$2x+1=\frac{\partial v}{\partial y}.$$

To get v we integrate both sides of this equation with respect to y. However, since v is a function of x and y, the constant of integration may be a function of x. Thus integrating with respect to y yields

$$v(x,y)=(2x+1)y+c(x),$$