Chapter 6
Harmonic Functions and Applications

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way.
-Godfrey Harold Hardy (1877-1947)

There are many important applications of complex analysis to real-world problems. The ones studied in this chapter are related to the fundamental differential equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

known as **Laplace's equation**. This partial differential equation models phenomena in engineering and physics, such as steady-state temperature distributions, electrostatic potentials, and fluid flow, just to name a few. A real-valued function that satisfies Laplace's equation is said to be harmonic. There is an intimate relationship between harmonic and analytic functions. This is investigated in Section 6.1 along with other fundamental properties of harmonic functions.

To illustrate an application, consider a two-dimensional plate of homogeneous material, with insulated lateral surfaces. We represent this plate by a region Ω in the complex plane (see Figure 6.1). Suppose that the temperature of the points on the boundary of the plate is described by the function $b(x, y)$ that does not change with time. It is a fact of thermodynamics that the temperature inside the plate will eventually reach and remain at an equilibrium distribution $u(x, y)$, known as the **steady-state temperature distribution**. which satisfies the equation $\Delta u = 0$.

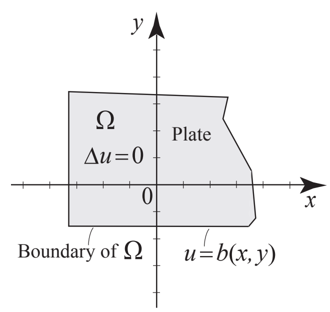

Fig. 6.1 The steady-state temperature distribution of a plate satisfies Laplace's equation.

The **Laplacian Δ** is named after the great French mathematician and physicist Pierre-Simon de Laplace (1749–1827). This operator appeared for the first time in a memoir of Laplace in 1784, in which he completely determined the attraction of a spheroid on the points outside it. The Laplacian of a function measures the difference between the value of the function at a point and the average value of the function in a neighborhood of that point. Thus a function that does not vary abruptly has a very small Laplacian. Harmonic functions have a zero Laplacian; they vary in a very regular way. Examples of such functions include the temperature distribution