and (5.7.15) follows from the formula for the Taylor coefficients.

33. Let a be an arbitrary complex number. Consider the equation $z = a + we^z$. Show that a solution of this equation is

$$z = a + \sum_{n=1}^{\infty} \frac{n^{n-1} e^{na}}{n!} w^n,$$

when $|w| < e^{-1-\operatorname{Re} a}$. [Hint: Let $z_0 = a$, $w = (z-a)e^{-z}$, $\phi(z) = \frac{z-a}{(z-a)e^{-z}} = e^z$, and apply Lagrange's inversion formula (5.7.15).]

34. Lambert's *w*-function. This function has been applied in quantum physics, fluid mechanics, biochemistry, and combinatorics. It is named after the German mathematician Johann Heinrich Lambert (1728–1777). The **Lambert function** or **Lambert *w*-function** is defined as the inverse function of $f(z) = ze^z$. Using the technique of Exercise 32, based on Lagrange's formula, show that the solution of $w = ze^z$ is

$$z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!} w^n \quad \text{whenever } |w| < \frac{1}{e}.$$

35. Project Problem: Landau's estimate. In this exercise we present Landau's solution of the following problem: Given an analytic function f in a neighborhood of a closed disk $\overline{B_R(z_0)}$ with $f'(z_0) \neq 0$, find $r > 0$ such that f is one-to-one on the open disk $B_r(z_0)$. Landau's solution: It suffices to take $r = R^2 |f'(z_0)|/(4M)$, where M is the maximum value of $|f|$ on $C_R(z_0)$.

Fill in the details in the following argument. It suffices to choose r so that $f'(z) \neq 0$ for all z in $B_r(z_0)$ (why?). Without loss of generality we can take $z_0 = 0$ and $w_0 = f(z_0) = 0$ (why?). Then for $|z| < R$, $f(z) = a_1 z + a_2 z^2 + \cdots$. Write $r = \lambda R$, where $0 < \lambda < 1$ is to be determined so that $f'(z) \neq 0$ for all z in $B_{\lambda R}(0)$. For z_1 and z_2 in $B_{\lambda R}(0)$, we have

$$\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| = \left| a_1 + \sum_{n=2}^{\infty} a_n \left(z_1^{n-1} + z_1^{n-2} z_2 + \cdots + z_1 z_2^{n-2} + z_2^{n-1} \right) \right| \geq |a_1| - \sum_{n=2}^{\infty} n |a_n| \lambda^{n-1} R^{n-1}.$$

If we could choose a number λ such that

$$\sum_{n=2}^{\infty} n |a_n| \lambda^{n-1} R^{n-1} < |a_1|, \tag{5.7.16}$$

this would make the absolute value of the difference quotient for the derivative positive, independently of z_1 and z_2 in $B_{\lambda R}(0)$, and this would in turn imply that $f'(z) \neq 0$ for all z in $B_{\lambda R}(0)$ and complete the proof. So let us show that we can choose λ so that (5.7.16) holds. Let $M = \max_{|z|=R} |f(z)|$. Cauchy's estimate yields $|a_n| \leq \frac{M}{R^n}$. Then

$$\sum_{n=2}^{\infty} n |a_n| \lambda^{n-1} R^{n-1} \leq \frac{M}{R} \sum_{n=2}^{\infty} n \lambda^{n-1} = \frac{M}{R} \frac{\lambda(2-\lambda)}{(1-\lambda)^2} < \frac{M}{R} \frac{2\lambda}{(1-\lambda)^2}.$$

(Use $\sum_{n=2}^{\infty} n\lambda^{n-1} = \frac{d}{d\lambda}(\lambda^2 + \lambda^3 + \cdots) = \frac{d}{d\lambda} \frac{\lambda^2}{1-\lambda} = \frac{2\lambda - \lambda^2}{(1-\lambda)^2}$.) Consider the choice $\lambda = \frac{R|a_1|}{4M}$. This yields $\lambda \leq \frac{1}{4}$ and $\sum_{n=2}^{\infty} n|a_n| \lambda^{n-1} R^{n-1} < \frac{|a_1|}{4\lambda} \frac{2\lambda}{(1-\lambda)^2} = \frac{|a_1|}{2} \frac{1}{(1-\lambda)^2} \leq \frac{8}{9} |a_1|$. (The maximum of $1/(1-\lambda)^2$ on the interval $[0, 1/4]$ occurs at $\lambda = 1/4$ and is equal to $\frac{16}{9}$.) Hence (5.7.16) holds for this choice of λ, and for this choice, we get $r = \lambda R = \frac{R^2 |a_1|}{4M}$, which is what Landau's estimate says.