Hurwitz's theorem has many interesting applications. We start with some theoretical properties and then give some applications to counting zeros of analytic functions (Exercises 28–30).

27. Suppose that f_n converges to f uniformly on every closed and bounded subset of a region Ω with f_n analytic and vanishing nowhere on Ω. Then either $f = 0$ identically or f has no zeros in Ω.

28. Univalent functions. (a) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of univalent functions on a region Ω that converges to f uniformly on every closed and bounded subset of Ω, and f is not identically constant on Ω. Then f is univalent. [Hint: Fix z_0 in Ω and apply Exercise 26 to the sequence of functions $\{f_n - f_n(z_0)\}_{n=1}^{\infty}$ defined on $\Omega \setminus \{z_0\}$.]
(b) Give an example of a sequence of univalent functions converging uniformly on the closed unit disk to a constant function.

29. Counting zeros with Hurwitz's theorem. If we want to find the number of zeros inside the unit disk of the polynomial $p(z) = z^5 + z^4 + 6z^2 + 3z + 1$, then we cannot just apply Rouché's theorem since there is not one single coefficient of the polynomial whose absolute value dominates the sum of the absolute values of the other coefficients. Here is how we can handle this problem:
(a) Consider the polynomials $p_n = p - \frac{1}{n}$. Show that p_n converges to p uniformly on the closed unit disk.
(b) Apply Rouché's theorem to show that p_n has two zeros in the unit disk.
(c) Apply Hurwitz's theorem to show that p has two zeros inside the unit disk.

In Exercises 30–31*, modify the steps in Exercise* 29 *to find the zeros of the polynomials in the indicated region.*

30. $z^5 + z^4 + 6z^2 + 3z + 11, \quad |z| < 1.$

31. $4z^4 + 6z^2 + z + 1, \quad |z| < 1.$

32. Project Problem: Lagrange's inversion formula. In this exercise, we outline a proof of the following useful inversion formula for analytic functions due to the French mathematician Joseph-Louis Lagrange (1736–1813).

Suppose that f is analytic at z_0, $w_0 = f(z_0)$, and $f'(z_0) \neq 0$, and let

$$\phi(z) = \frac{z - z_0}{f(z) - w_0}, \quad z \neq z_0. \tag{5.7.14}$$

Show that the inverse function $z = g(w)$ has a power series expansion

$$g(w) = z_0 + \sum_{n=1}^{\infty} b_n (w - w_0)^n, \quad \text{where} \quad b_n = \frac{1}{n!} \left. \frac{d^{n-1}}{dz^{n-1}} [\phi(z)]^n \right|_{z=z_0}. \tag{5.7.15}$$

Fill in the details in the following proof. Note that ϕ is analytic at z_0 (why?). To prove (5.7.15), start with the formula for the inverse function (5.7.12) and differentiate with respect to w using Theorem 3.8.5 and then integrate by parts to obtain

$$\begin{aligned} g'(w) &= \frac{1}{2\pi i} \int_{C_R(z_0)} z \frac{f'(z)}{(f(z) - w)^2} \, dz \\ &= -\frac{1}{2\pi i} \int_{C_R(z_0)} z \, d((f(z) - w)^{-1}) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{dz}{f(z) - w}. \end{aligned}$$

Differentiate under the integral sign $n - 1$ more times and evaluate at $w = w_0$ to obtain

$$g^{(n)}(w_0) = \frac{(n-1)!}{2\pi i} \int_{C_R(z_0)} \frac{dz}{(f(z) - w_0)^n} = \frac{(n-1)!}{2\pi i} \int_{C_R(z_0)} \phi(z)^n \frac{dz}{(z - z_0)^n}.$$

Hence by Cauchy's generalized integral formula

$$g^{(n)}(w_0) = \left. \frac{d^{n-1}}{dz^{n-1}} \phi(z)^n \right|_{z=z_0},$$