19. $\displaystyle\int_{C_2(0)} \frac{ze^{iz}}{z^2+1}\,dz$ **20.** $\displaystyle\int_{C_1(0)} \frac{z^3 e^{z^2}}{e^{z^2}-1}\,dz$

21. Summing roots of unity. We use the variant of the counting theorem to show that for $n \geq 2$, the sum of the n nth roots of unity is 0 (see Exercise 63, Section 1.3). Let S denote this sum.
(a) Using Theorem 5.7.6, explain why

$$S = \frac{1}{2\pi i}\int_{C_R(0)} z\frac{nz^{n-1}}{z^n-1}\,dz = \frac{n}{2\pi i}\int_{C_{1/R}(0)} \frac{1}{1-z^n}\frac{dz}{z^2},$$

where $R > 1$. [Hint: To prove the second equality make a suitable change of variables.]
(b) Evaluate the second integral in part (a) using Cauchy's generalized integral formula and conclude that $S = 0$.

22. Examples concerning Lemma 5.7.1. Give an example of a function f with an essential singularity at 0 such that $\frac{f'}{f}$ has
(a) an essential singularity at 0;
(b) a pole of order $m \geq 2$ at 0.
[Hint: Use suitable compositions of the function $e^{1/z}$ in your examples.]

23. Minimum modulus principle. Show that for a nonzero nonconstant analytic function f on a region Ω, $|f|$ does not attain a minimum in Ω. [Hint: Use that the function f is open.]

24. Meromorphic Rouché's theorem. Suppose that C is a simple closed path, Ω is the region inside C, and f and g are meromorphic inside and on C, having no zeros or poles on C. Show that if $|g(z)| < |f(z)|$ for all z on C, then $N(f+g) - P(f+g) = N(f) - P(f)$. [Hint: Repeat the proof of Rouché's theorem. What can you say about the values of ϕ in the present case?]

25. Complete the argument that provides a geometric proof of Theorem 5.7.9 (Rouché's theorem).
(i) Show that for each $z \in C$, we can find a branch of the argument where

$$\arg f(z) - \frac{\pi}{2} < \arg\left(f(z)+g(z)\right) < \arg f(z) + \frac{\pi}{2}.$$

(ii) Using connectedness, we can show that this inequality holds for the specific argument function $\arg f$ used to define $\Delta_C \arg f$, see page 353. Show that

$$\Delta_C \arg f - \pi < \Delta_C \arg\left(f+g\right) < \Delta_C \arg f + \pi,$$

and use the fact that $\Delta_C \arg\left(f+g\right)$ must be an integer multiple of 2π to prove that $\Delta_C \arg f = \Delta_C \arg\left(f+g\right)$ on C.

26. Project Problem: Hurwitz's theorem. We outline a proof of a useful theorem due to the German mathematician Adolf Hurwitz (1859–1919). The theorem states the following: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of analytic functions on a region Ω converging uniformly on every closed and bounded subset of Ω to a function f. Then either
(i) f is identically 0 on Ω; or
(ii) if $B_r(z_0)$ is an open disk in Ω such that f does not vanish on $C_r(z_0)$, then f_n and f have the same number of zeros in $B_r(z_0)$ for all sufficiently large n. In particular, if f is not identically 0 and f has p distinct zeros in Ω, then so do the functions f_n for all sufficiently large n.

Observe that f is analytic by Theorem 4.1.10. Also, note that the theorem guarantees that, for large n, f_n and f have the same number of zeros, but these zeros are not necessarily the same for f_n and f. To see this, take $f_n(z) = z - \frac{1}{n}$ and $f(z) = z$ for all $z \in \Omega$. Finally, observe that the possibility that f is identically zero can arise, even if the f_n's are all nonzero. Simply take $f_n = \frac{1}{n}$.

Fill in the details in the following proof. Suppose that f is not identically 0 in Ω. Let $\overline{B_r(z_0)}$ be a closed disk such that f is nonvanishing on $C_r(z_0)$. Let $m = \min|f|$ on $C_r(z_0)$. Then $m > 0$ (why?). Apply uniform convergence to get an index N such that $n > N$ implies that $|f_n - f| < m \leq |f|$ on $C_r(z_0)$. Complete the proof by applying Rouché's theorem.