we have the following result, which should be compared to Theorem 2.3.12. Theorem 5.7.17 can be used to derive a useful formula due to Lagrange for the inversion of power series. See Exercise 32.

A function f that is analytic and one-to-one is called a **univalent** function. The next corollary says that univalent functions have global inverses.

Corollary 5.7.18. (A Global Inverse Function) *Suppose that f is univalent function on a region Ω. Then its inverse function f^{-1} exists and is analytic on the region $f[\Omega]$. Moreover,*

$$\frac{d}{dw}f^{-1}(w) = \frac{1}{f'(z)}, \quad \text{where } w = f(z). \tag{5.7.13}$$

Proof. By Theorem 5.7.17, f^{-1} is analytic in a neighborhood of each point in Ω, and hence it is analytic on Ω. Also, since f is one-to-one, Theorem 5.7.13 implies that $f'(z) \neq 0$ for all z in Ω. Differentiating both sides of the identity $z = f^{-1}(f(z))$, we obtain $1 = \frac{d}{dw}f^{-1}(w)f'(z)$, which is equivalent to (5.7.13). ■

Exercises 5.7

In Exercises 1–6, use the method of Example 5.7.5 to find the number of zeros in the first quadrant of the following polynomials.

1. z^2+2z+2

2. z^2-2z+2

3. z^3-2z+4

4. z^3+5z^2+8z+6

5. $z^4+8z^2+16z+20$

6. $z^5+z^4+13z^3+10$

In Exercises 7–14, use Rouché's theorem to determine the number of zeros of the functions in the indicated region.

7. $z^3+3z+1, \quad |z|<1$

8. $z^4+4z^3+2z^2-7, \quad |z|<2$

9. $7z^3+3z^2+11, \quad |z|<1$

10. $7z^3+z^2+11z+1, \quad 1<|z|$

11. $4z^6+41z^4+46z^2+9, \quad 2<|z|<4.$

12. $z^4+50z^2+49, \quad 3<|z|<4$

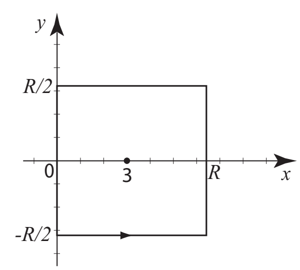
13. $e^z-3z, \quad |z|<1$

14. $e^{z^2}-4z^2, \quad |z|<1$

15. Show that the equation

$$3-z+2e^{-z}=0$$

has exactly one root in the right half-plane $\operatorname{Re} z>0$. [Hint: Use Rouché's theorem and contours such as the one in the adjacent figure.]



16. Suppose that $\operatorname{Re} w>0$ and let a be a complex number. Show that $w-z+ae^{-z}=0$ has exactly one root in the right half-plane $\operatorname{Re} z>0$.

In Exercises 17–20, evaluate the path integrals. As usual, $C_R(z_0)$ stands for the positively oriented circle with radius $R>0$ centered at z_0.

17. $\displaystyle\int_{C_1(0)} \frac{dz}{z^5+3z+5}$

18. $\displaystyle\int_{C_1(0)} \frac{e^z-12z^3}{e^z-3z^4}\,dz$