$f^{-1}[U]$ and $f^{-1}[V]$ are open (note that since f is defined on Ω, when taking a pre-image, we only consider those points in Ω). Clearly, $f^{-1}[U]$ and $f^{-1}[V]$ are disjoint and their union is Ω. Since Ω is connected, either $\Omega = f^{-1}[U]$ or $\Omega = f^{-1}[V]$, and hence $f[\Omega] = U$ or $f[\Omega] = V$, implying that $f[\Omega]$ is connected. ∎

Another interesting application of the open mapping property of analytic functions is a simple proof of the maximum principle.

Corollary 5.7.16. (Maximum Modulus Principle) *If f is analytic on a region Ω, such that $|f|$ attains a maximum at some point in Ω, then f is constant in Ω.*

Proof. Suppose f is nonconstant. Let z_0 be an arbitrary point in Ω; we will show that $|f(z_0)|$ is not a maximum. By Corollary 5.7.14, f is open. Then for a small open disk $B_\varepsilon(z_0)$ the set $f[B_\varepsilon(z_0)]$ is open and thus it contains an open disk $B_\rho(w_0)$ of $w_0 = f(z_0)$. Clearly there exists w in $B_\rho(w_0)$ such that $|w| > |w_0|$. See Figure 5.63. Then $w = f(z)$ for some $z \in B_\varepsilon(z_0)$ and $|f(z)| > |f(z_0)|$. Consequently $f(z_0)$ does not have the largest modulus among all the points in $B_\rho(w_0)$ and thus $|f(z_0)|$ cannot be a maximum of $|f|$. ∎

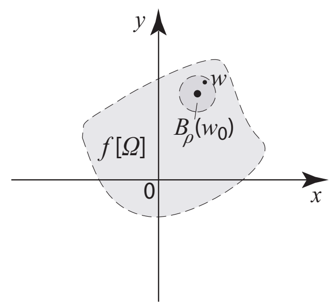

Fig. 5.63 $B_\rho(w_0) \subset f[\Omega]$.

We next use the variant of the counting theorem to give a formula for the inverse function of f in terms of an integral involving f.

Theorem 5.7.17. (Inverse Function Formula) *Suppose that f is analytic in a neighborhood of a point z_0 and $f'(z_0) \neq 0$. Let $w_0 = f(z_0)$. Then there are $R, \rho > 0$ such that $f^{-1}[B_\rho(w_0)] \subset B_R(z_0)$, f is one to one on $f^{-1}[B_\rho(w_0)]$, and has an inverse function $g : B_\rho(w_0) \to f^{-1}[B_\rho(w_0)]$ which is is analytic on $B_\rho(w_0)$ and satisfies*

$$g(w) = \frac{1}{2\pi i} \int_{C_R(z_0)} z \frac{f'(z)}{f(z) - w} \, dz, \qquad w \in B_\rho(w_0). \tag{5.7.12}$$

Proof. Let R and ρ be as in Theorem 5.7.12. For w in $B_\rho(w_0)$, the function $f - w$ has exactly one zero in $B_R(z_0)$, which we call $g(w)$. Applying identity (5.7.7) (with $n_1 = 1$, $z_1 = g(w)$, $m(z_1) = 1$, $n_2 = 0$) we see that identity (5.7.12) holds. **DELETE BLUE TEXT:** Then g is continuous on $B_\rho(w_0)$, as for $w \to w_1$ we can pass the limit inside the integral in (5.7.12) to obtain $g(w) \to g(w_1)$. Applying Theorem 2.3.12 [with $h = f \circ g$ being the identity map on $B_\rho(w_0)$] we obtain that g is analytic on $B_\rho(w_0)$. The analyticity of the function g is a consequence of Theorem 3.8.5. ∎

Theorem 5.7.17 can be viewed as a statement about the local existence of inverse functions. If f is one-to-one on a region Ω, there is no ambiguity in defining the inverse function f^{-1} on the whole region $f[\Omega]$. Indeed, for w in $f[\Omega]$, we define $f^{-1}(w)$ to be the unique z in Ω with $f(z) = w$. For f analytic and one-to-one,