$B_\rho(w_0)$ under f. Since $B_\rho(w_0)$ is open and f is continuous, U is open (Exercise 41, Section 2.2) and it clearly contains z_0. So we can find an open disk $B_r(z_0)$ that is contained in U and $B_R(z_0)$. Then f is one-to-one on $B_r(z_0)$, for if $f(z_1) = f(z_2)$ is a point in $B_\rho(w_0)$, Theorem 5.7.12 guarantees $z_1 = z_2$. Conversely, if $f'(z_0) = 0$, then the order of f at z_0 is at least 2. Applying Theorem 5.7.12 we find a neighborhood of z_0 on which f is at least two-to-one; so f is not one-to-one in this case. ■

A few comments are in order regarding the inverse function theorem. The theorem can be obtained from the classical inverse function theorem in two variables. The latter states that the mapping $(x, y) \mapsto (u(x, y), v(x, y))$ is one-to-one in a neighborhood of (x_0, y_0) if its Jacobian is nonzero at (x_0, y_0); recall that the Jacobian of this mapping at (x, y) is

$$J(x, y) = \det \begin{vmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{vmatrix} = u_x(x, y)v_y(x, y) - u_y(x, y)v_x(x, y).$$

Hence, using the Cauchy-Riemann equations, we find that

$$J(x, y) = u_x^2(x, y) + u_y^2(x, y) = |f'(x + iy)|^2.$$

So if $f'(x_0 + iy_0) \neq 0$, then $J(x_0, y_0) \neq 0$ and Theorem 5.7.13 follows from the inverse function theorem for functions of two variables as claimed.

It is important to keep in mind that the condition $f'(z) \neq 0$ for all z in Ω does not imply that f is one-to-one on Ω. Consider $f(z) = e^z$; then $f'(z) = e^z \neq 0$ for all z, yet f is not one-to-one on the whole complex plane. Theorem 5.7.13 only guarantees that f is one-to-one in *some* neighborhood of a point z_0 where $f'(z_0) \neq 0$. Because this neighborhood depends on z_0 in general, an obvious question is whether we can estimate its size in terms of the sizes of f and f'. An answer to this question was provided by the German mathematician Edmund Landau (1877–1938) and is known as **Landau's estimate**. See Exercise 35.

The following two corollaries describe important properties of analytic functions, which are direct applications of the local mapping theorem.

Corollary 5.7.14. (Open Mapping Property) *Let Ω be a region. Then for a nonconstant analytic function f on Ω and an open subset U of Ω the set $f[U]$ is open. A function with this property is said to be* **open***.*

Proof. Let U be an open subset of Ω. Given w_0 in $f[U]$, let z_0 be some point in U where $f(z_0) = w_0$. Since U is open, we can find a neighborhood $B_R(z_0)$ contained in Ω; by applying Theorem 5.7.12 to $B_R(z_0)$, we find that each point in the associated $B_\rho(w_0)$ is assumed by f. Thus $B_\rho(w_0)$ is contained in $f[U]$, and $f[U]$ is open. ■

Corollary 5.7.15. (Mapping of Regions) *If Ω is a nonempty region (open and connected set) and f is a nonconstant analytic function on Ω, then $f[\Omega]$ is a region.*

Proof. By Corollary 5.7.14, $f[\Omega]$ is open. To show that $f[\Omega]$ is connected, suppose that $f[\Omega] = U \cup V$, where U and V are open and disjoint. Since f is continuous,