(b) Take $f(z) = az^n$ and $g(z) = -e^z$. Counted according to multiplicity, f has n zeros in $|z| < 1$ and so $N(f) = n$. For $|z| = 1$, $|f(z)| = |az^n| = a$ and

$$|g(z)| = |e^z| = \left|e^{\cos\theta + i\sin\theta}\right| = e^{\cos\theta} \le e < a.$$

Thus $|g(z)| < |f(z)|$ for all $|z| = 1$ and so by Rouché's theorem $n = N(az^n) = N(az^n - e^z)$, which implies that $az^n - e^z = 0$ has n roots in $|z| < 1$. □

As a further application we give a very simple proof of the fundamental theorem of algebra.

Example 5.7.11. (The fundamental theorem of algebra) Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_n \neq 0$ be a polynomial of degree $n \ge 1$. Show that p has exactly n roots, counting multiplicity.

Solution. Take $f(z) = a_n z^n$. Then f has a zero of multiplicity n at $z = 0$. Also, for $|z| = R$, we have $|f(z)| = |a_n| R^n$, which is a polynomial of degree n in R. Now let $g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, then $|g(z)| \le |a_{n-1}|\,|z^{n-1}| + \cdots + |a_1|\,|z| + |a_0|$, and so for $|z| = R$, $|g(z)| \le |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0|$. Since the modulus of f grows at a faster rate than the modulus of g, in the sense that

$$\lim_{R\to\infty} \frac{1}{|a_n| R^n} \left(|a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0|\right) = 0,$$

we can find R_0 large enough so that for $R \ge R_0$, we have

$$|a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0| < |a_n| R^n.$$

This implies that $|g(z)| < |f(z)|$ for all $|z| = R$ with $R \ge R_0$. By Rouché's theorem, $N(f)$, the number of zeros of f in the region $|z| < R$, is the same as $N(f+g)$, the number of zeros of $f+g$. But $N(f) = n$ and $f + g = p$, so $N(p) = n$ showing that p has exactly n zeros. □

The Local Mapping Theorem

In this part, we investigate fundamental properties of the inverse function of analytic functions. When do they exist? Are they analytic? Are there explicit formulas for them? All these questions can be answered with the help of Rouché's theorem and the counting theorem. Our investigation leads to interesting new properties of analytic functions and shed new light on some classical results studied earlier. In particular, we give a simple proof of the maximum modulus principle. Interesting applications of these topics are presented in the exercises, including a formula due