Proof. For z on C, the inequality $|g(z)| < |f(z)|$ implies that $f(z) \neq 0$, and so $|g|/|f|$ is continuous and strictly less than 1 on C. Since C is a closed and bounded set, $|g|/|f|$ attains its maximum $\delta < 1$ on C. Pick $\delta < \delta_1 < 1$. For z on C and $0 < |\lambda| \leq 1/\delta_1$, if $f(z) + \lambda g(z) = 0$, then $|g(z)|/|f(z)| = 1/|\lambda| \geq \delta_1$, which contradicts that $|g|/|f| \leq \delta$ on C. So $f(z) + \lambda g(z)$ is not equal to zero for all z on C and $|\lambda| \leq 1/\delta_1$. We claim that the function

$$\phi(\lambda) = \frac{1}{2\pi i} \int_C \frac{f'(z) + \lambda g'(z)}{f(z) + \lambda g(z)}\, dz \tag{5.7.8}$$

is continuous in λ on $[-1/\delta_1, 1/\delta_1]$. To verify this, notice that the integrand $G(\lambda, z)$ in (5.7.8) is continuous and thus uniformly continuous on $[0, 1/\delta_1] \times C$. Consequently, $G(\lambda, z)$ is close to $G(\lambda', z)$ uniformly in z and thus the integral of $G(\lambda, z)$ in z is close to that of $G(\lambda', z)$ in z. By Theorem 5.7.2, $\phi(\lambda) = N(f + \lambda g)$, and so ϕ is integer-valued. By Lemma 5.7.8, ϕ is constant for all $|\lambda| < 1/\delta_1$. In particular, $\phi(0) = N(f) = \phi(1) = N(f+g)$. ■

We can give a geometric interpretation of Rouché's theorem, based on the argument principle. According to (5.7.6), we are merely claiming that

$$\Delta_C \arg(f+g) = \Delta_C \arg f. \tag{5.7.9}$$

As z traces C, $f(z)$ winds around the origin a specific number of times. Since $|g(z)| < |f(z)|$, the point $f(z) + g(z)$ must lie in the disk of radius $|f(z)|$ centered at $f(z)$ (Figure 5.62), and so $f(z) + g(z)$ must wind around the origin the same number of times as $f(z)$ (see Exercise 25).

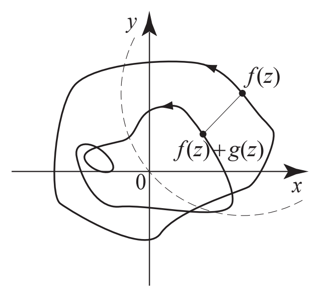

Fig. 5.62 Suppose that for each z on C there is a rope, shorter than the distance from $f(z)$ to the origin, joining $f(z)$ to $f(z)+g(z)$. Then $f(z)+g(z)$ goes around the origin the same number of times as $f(z)$ goes around the origin, when z traces the curve C.

The following are typical applications of Rouché's theorem.

Example 5.7.10. (Counting zeros with Rouché's theorem)
(a) Show that all zeros of $p(z) = z^4 + 6z + 3$ lie inside the circle $|z| = 2$.
(b) Show that if a is a real number with $a > e$, then in $|z| < 1$ the equation $e^z = az^n$ has n roots (counting orders).

Solution. (a) Since p is a polynomial of degree 4, it is enough to show that $N(p) = 4$ inside the circle $|z| = 2$. Take $f(z) = z^4$ and $g(z) = 6z + 3$ and note that $f(z) + g(z) = p(z)$. For $|z| = 2$, we have $|f(z)| = 2^4 = 32$ and $|g(z)| \leq |6z| + 3 = 15$. Hence $|g(z)| < |f(z)|$ for all z on the circle $|z| = 2$, and so by Rouché's theorem $N(f) = N(f+g) = N(p)$. Clearly f has one zero with multiplicity 4 at $z = 0$. Thus $N(f) = 4$ and so $N(p) = 4$, as desired.