$$\operatorname{Res}\left(g\frac{f'}{f}, p_j\right) = g(p_j)\operatorname{Res}\left(\frac{f'(z)}{f(z)}, p_j\right) = -m(p_j)g(p_j).$$

Now (5.7.7) follows from the residue theorem. ■

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**Example 5.7.7.** Evaluate

$$\int_{C_1(0)} \frac{e^z \cos z}{e^z - 1}\, dz,$$

where $C_1(0)$ is the positively oriented unit circle.

**Solution.** The function $f(z) = e^z - 1$ has a zero at $z = 0$ and, because $f'(0) = 1 \neq 0$, this zero is simple. Also, using the $2\pi i$-periodicity of $e^z$, it is easy to see that $e^z = 1$ inside the unit disk $|z| < 1$ only at $z = 0$. Applying (5.7.7) with $f(z) = e^z - 1$ and $g(z) = \cos z$ it follows immediately that

$$\int_{C_1(0)} \frac{e^z \cos z}{e^z - 1}\, dz = 2\pi i \cos 0 = 2\pi i.$$ □

As an application of the counting principle we derive a famous result known as Rouché's theorem, named after the French mathematician and educator Eugène Rouché (1832–1910). We need the following lemma.

**Lemma 5.7.8.** *Let $\phi$ be a continuous function on a region $\Omega$ that takes only integer values. Then $\phi$ is constant in $\Omega$.*

*Proof.* Assume that $\phi$ is not constant, and let $z_1$ and $z_2$ in $\Omega$ be such that $\phi(z_1) = n_1 < \phi(z_2) = n_2$. Let $r$ be a real number such that $n_1 < r < n_2$. Since $\phi$ is continuous, the sets

$$A = \{z \in \Omega : \ \phi(z) < r\}$$
$$B = \{z \in \Omega : \ \phi(z) > r\}$$

are open. Also, $z_1 \in A$ and $z_2 \in B$, hence $A$ and $B$ are nonemtpy. They are also disjoint and satisfy $A \cup B = \Omega$. This contradicts the fact that $\Omega$ is connected. Thus $\phi$ is constant. ■

In what follows, we need a version of Lemma 5.7.8, where $\phi$ is a continuous, integer-valued function on an interval $[a, b]$. The preceding proof can be easily modified to cover this case.

**Theorem 5.7.9. (Rouché's Theorem)** *Suppose that $C$ is a simple closed path, $\Omega$ is the region inside $C$, $f$ and $g$ are analytic inside and on $C$. If $|g(z)| < |f(z)|$ for all $z$ on $C$, then*

$$N(f+g) = N(f)$$

*in other words, $f+g$ and $f$ have the same number of zeros ~~on~~ in $\Omega$.*