clearly positive. If $z = iy$, with $y \geq 0$, then $f(iy) = (iy)^6 + 6iy + 10 = (-y^6 + 10) + 6iy$ and because $\operatorname{Im}(f(iy)) = 0 \Leftrightarrow y = 0 \Rightarrow \operatorname{Re}(f(iy)) \neq 0$, we see that $f(iy) \neq 0$.
Step 2: Compute the change of the argument of $f(z)$ as z varies from the initial to the terminal point of $I_1 = [0, R]$. For $z = 0$, $f(z) = 10$, and for $0 \leq z = x \leq R$, $f(z) = x^6 + 6x + 10 > 0$. So the image of the interval $[0, R]$ is the interval $[10, R^6 + 6R + 10]$ and the argument of $f(z)$ does not change on I_1.
Step 3: Compute the change of the argument of $f(z)$ as z varies from the initial to the terminal point of the arc γ_R. Here we are not looking for the exact image of γ_R by f, but only a rough picture that gives us the change in the argument of f. For very large R and z on γ_R, write $z = Re^{i\theta}$, where $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$f(z) = R^6 e^{6i\theta}\left(1 + \frac{6}{R^5}e^{-i5\theta} + \frac{10}{R^6}e^{-6i\theta}\right) \approx R^6 e^{6i\theta},$$

because $\frac{6}{R^5}e^{-i5\theta} + \frac{10}{R^6}e^{-6i\theta} \approx 0$. So as θ varies from 0 to $\frac{\pi}{2}$, the argument of $f(Re^{i\theta})$ varies from 0 to $6 \cdot \frac{\pi}{2} = 3\pi$. In fact, the point $f(iR) = -R^6 + 10 + 6iR$ lies in the second quadrant and has argument very close to 3π. See Figure 5.61.
Step 4: Compute the change of the argument of $f(z)$ as z varies from iR to 0. As z varies from iR to 0, $f(z)$ varies from $w_3 = -R^6 + 10 + 6iR$ to $w_0 = 10$. Since $\operatorname{Im} f(z) \geq 0$, this tells us that the point $f(z)$ remains in the upper half-plane as $f(z)$ moves from w_3 to w_0. Hence the change in the argument of $f(z)$ is $-\pi$.
Step 5: Apply the argument principle. The net change of the argument of $f(z)$ as we travel once around C is $3\pi - \pi = 2\pi$. According to (5.7.6), the number of zeros of f inside C, and hence in the first quadrant, is $\frac{1}{2\pi}2\pi = 1$. □

We give one more version of the counting theorem.

Theorem 5.7.6. (Variant of the Counting Theorem) *Let C, Ω, and f be as in Theorem 5.7.3, let g be analytic on an open set that contains C and its interior. Let $z_1, z_2, \ldots, z_{n_1}$ denote the zeros of f in Ω and $p_1, p_2, \ldots, p_{n_2}$ denote the poles of f in Ω. Let $m(z_j)$ be the order of the ~~root~~ zero z_j of f and $m(p_j)$ denote the order of the pole p_j of f. Then*

$$\frac{1}{2\pi i}\int_C g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n_1} m(z_j)g(z_j) - \sum_{j=1}^{n_2} m(p_j)g(p_j). \tag{5.7.7}$$

Proof. We modify the proof of the previous theorem as follows. If z_j is a zero of f, then, since g is analytic and $\frac{f'}{f}$ has a simple pole at z_j, using Lemma 5.7.1 and Proposition 5.1.3, we obtain

$$\begin{aligned}\operatorname{Res}\left(g\frac{f'}{f}, z_j\right) &= \lim_{z\to z_j}(z - z_j)g(z)\frac{f'(z)}{f(z)} = g(z_j)\lim_{z\to z_j}(z - z_j)\frac{f'(z)}{f(z)} \\ &= g(z_j)\operatorname{Res}\left(\frac{f'}{f}, z_j\right) = g(z_j)m(z_j).\end{aligned}$$

Similarly for the poles p_j,