A function is called **meromorphic** on a region if it is analytic in this region except at its poles. For example, $\frac{\sin z}{z^2+1}$ is meromorphic in the complex plane. Like zeros, we will count poles according to multiplicity. Next we generalize Theorem 5.7.2 to meromorphic functions.

Theorem 5.7.3. (Meromorphic Counting Theorem) *Suppose that C is a simple closed positively oriented path, Ω is the region inside C, and f is meromorphic on Ω and analytic and nonvanishing on C. Let $N(f)$ denote the number of zeros of f inside Ω and $P(f)$ denote the number of poles of f inside Ω, counted according to multiplicity. Then $N(f)$ and $P(f)$ are finite and*

$$N(f)-P(f)=\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz. \tag{5.7.4}$$

Proof. That $N(f)$ and $P(f)$ are finite follows as in the proof of Theorem 5.7.2. Applying Theorem 5.1.2 and using Lemma 5.7.1 we see that (5.7.4) holds. ∎

Either of the preceding theorems is also known as the **argument principle** because the right sides of (5.7.3) and (5.7.4) can be interpreted as the change in argument as one runs around the image path $f[C]$. We now investigate this.

Proposition 5.7.4. (Branch of the Logarithm) *If f is analytic and nonvanishing on a simply connected region Ω, then there exists an analytic branch of the logarithm,* $\log f=\ln|f|+i\arg f$, *such that for all z in Ω we have*

$$\frac{d}{dz}\log f(z)=\frac{f'(z)}{f(z)}\,. \tag{5.7.5}$$

Proof. We know from Corollary 3.6.9(ii) that every analytic function on a simply connected region Ω has an antiderivative. In particular, if f is analytic and nonvanishing on Ω, then f'/f is analytic on Ω and thus has an antiderivative g on Ω. We verify that the derivative of f/e^g is zero, hence by Theorem 2.5.7 we have that $f=ce^g$ for some nonzero complex constant c. Writing $c=e^d$, we obtain that $f=e^{g+d}$, where g is analytic and d is a constant.

We call the function $g+d$ a branch of the logarithm of f and write it as $\log f$. Indeed, the branch of the logarithm has the form

$$\log f=\ln|f|+i\arg f,$$

where $\arg f$ is a continuous branch of the argument. Then

$$e^{\log f}=f \qquad \text{hence} \qquad (\log f)'e^{\log f}=f',$$

which implies (5.7.5). ∎

The integrand in (5.7.4) suggests a connection with the logarithm of f, which we now explore. Let f and C be as in Theorem 5.7.3. Since $f[C]$ is a closed and bounded set that does not contain the origin, we can partition C into small subarcs γ_j $(j=1,\dots,n)$ such that each image $f[\gamma_j]$ is contained in a simply connected region that does not contain the origin, as shown in Figure 5.58.