$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} f(k) = -\pi \sum_j \operatorname{Res}\big(f(z)\cot(\pi z), z_j\big), \tag{5.6.6}$$

where the (finite) sum on the right runs over all the poles z_j of $f(z)$, including 0. Prove (5.6.6) by modifying the proof of Proposition 5.6.2; more specifically, explain what happens to (5.6.4) under the current conditions.

In Exercises 14–16*, use* (5.6.6) *to derive the identity.*

14. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

15. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$

16. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2(k^2+4)} = \frac{3+4\pi^2-6\pi\coth(2\pi)}{96}$

17. Project Problem: Sums of the reciprocals of even powers of integers. In this exercise, we use (5.6.6) to derive

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1}\frac{2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}, \tag{5.6.7}$$

where n is a positive integer, B_{2n} is the Bernoulli number (Example 4.3.12). This remarkable identity sums the reciprocals of the even powers of the integers. There is no known finite expression corresponding to any odd powers.

(a) Show that if $f(z) = \frac{1}{z^{2n}}$ then (5.6.6) becomes

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^{2n}} = -\pi \operatorname{Res}\Big(\frac{\cot(\pi z)}{z^{2n}}, 0\Big),$$

and so

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = -\frac{\pi}{2} \operatorname{Res}\Big(\frac{\cot(\pi z)}{z^{2n}}, 0\Big). \tag{5.6.8}$$

(b) Using the Taylor series expansion of $z\cot z$ from Exercise 31, Section 4.3, obtain

$$\operatorname{Res}\Big(\frac{\cot(\pi z)}{z^{2n}}, 0\Big) = (-1)^n \frac{2^{2n}B_{2n}\pi^{2n-1}}{(2n)!};$$

then derive (5.6.7).

18. Project Problem: Sums with alternating signs. (a) Modify the proof of Proposition 5.6.2 to prove the following summation result. Suppose that $f = \frac{p}{q}$ is a rational function with degree $q \geq 2 +$ degree p. Suppose further that f has no poles at the integers, except possibly at 0. Then,

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (-1)^k f(k) = -\pi \sum_j \operatorname{Res}\big(f(z)\csc(\pi z), z_j\big), \tag{5.6.9}$$

where the (finite) sum on the right is taken over all the poles z_j of f, including 0. [Hint: You need a version of Lemma 5.6.1 for the cosecant.]

(b) Show that if $f(z) = \frac{1}{z^{2n}}$ then (5.6.9) becomes

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k}{k^{2n}} = -\pi \operatorname{Res}\Big(\frac{\csc(\pi z)}{z^{2n}}, 0\Big),$$

and so