

13. (a) Use the method of Example 5.5.4 and the contour in Figure 5.51 to establish the identity

$$\text{P.V.} \int_0^\infty \frac{x^p}{x(1-x)} dx = \pi \cot p\pi \quad (0 < p < 1).$$

(b) Use a suitable change of variables to derive the identity

$$\text{P.V.} \int_{-\infty}^\infty \frac{e^{px}}{1-e^x} dx = \pi \cot p\pi \quad (0 < p < 1).$$

(c) Use (b) to show that for $-1 < w < 1$

$$\text{P.V.} \int_{-\infty}^\infty \frac{e^{wx}}{\sinh x} dx = \pi \tan \frac{\pi w}{2}.$$

(d) Use a suitable change of variables to show that

$$\text{P.V.} \int_{-\infty}^\infty \frac{e^{ax}}{\sinh bx} dx = \frac{\pi}{b} \tan \frac{\pi a}{2b} \quad (b > |a|).$$

(e) Conclude that

$$\int_{-\infty}^\infty \frac{\sinh ax}{\sinh bx} dx = \frac{\pi}{b} \tan \frac{\pi a}{2b} \quad (b > |a|).$$

Note that the integral is convergent so there is no need to use the principal value.

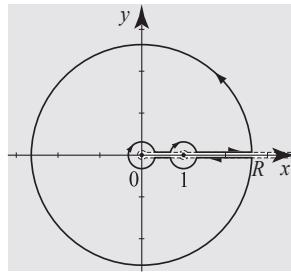


Fig. 5.51 Exercise 13.

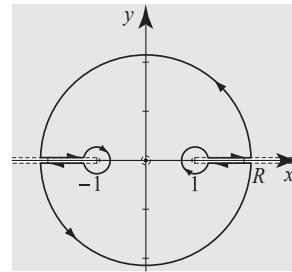


Fig. 5.52 Exercise 14.

14. Use the contour in Figure 5.52 to establish the identity

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{2}.$$

In Exercises 15–20, derive the identities.

$$15. \int_0^\infty \frac{dx}{(x+2)\sqrt{x+1}} = \frac{\pi}{2}$$

$$16. \int_0^\infty \frac{dx}{(x+2)^2\sqrt{x+1}} = \frac{\pi}{4} - \frac{1}{2}$$

$$17. \int_0^\infty \frac{\sqrt{x}}{x^2+x+1} dx = \frac{\pi}{\sqrt{3}}$$

$$18. \int_0^\infty \frac{x^a}{x^2+1} dx = \frac{\pi}{2} \sec \frac{a\pi}{2} \quad (-1 < a < 1)$$

$$19. \int_0^\infty \frac{x}{\sinh x} dx = \frac{\pi^2}{4}$$

$$20. \int_0^\infty \frac{dx}{\cosh x} = \frac{\pi}{2}$$

21. Integral of the Gaussian with complex parameters. We show that, for α and β complex numbers with $\operatorname{Re} \alpha > 0$,