Proof. Parametrize the integral in (5.4.20) by $z = z_0 + re^{i\theta}$, where $\theta_0 \le \theta \le \theta_0 + \alpha$, $dz = rie^{i\theta}\, d\theta$, $z - z_0 = re^{i\theta}$. Then

$$\begin{aligned}
&\frac{1}{i\alpha}\int_{\sigma_r} \frac{f(z)}{z-z_0}\,dz \\
&= \frac{1}{i\alpha}\int_{\theta_0}^{\theta_0+\alpha} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta \\
&= \frac{1}{\alpha}\int_{\theta_0}^{\theta_0+\alpha} f(z_0+re^{i\theta})\, d\theta.
\end{aligned}$$

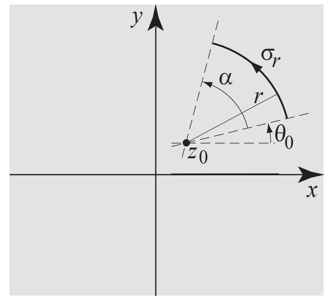

Fig. 5.29 Circular arc of angle α.

Define the function

$$F(r) = \frac{1}{\alpha}\int_{\theta_0}^{\theta_0+\alpha} f(z_0+re^{i\theta})\, d\theta.$$

Since f is continuous on $\overline{B_{r_0}(z_0)}$, it follows that it is uniformly continuous on this set. Thus given $\varepsilon > 0$ there is $\delta > 0$ such that $|w - w'| < \delta$ implies $|f(w) - f(w')| < \varepsilon$.

Then $|r - r'| < \delta$ implies $\big|(z_0 + re^{i\theta}) - (z_0 + r'e^{i\theta})\big| < \delta$ which in turn implies $|f(z_0 + re^{i\theta}) - f(z_0 + r'e^{i\theta})| < \varepsilon$. The *ML*-inequality yields

$$|F(r) - F(r')| = \left|\frac{1}{\alpha}\int_{\theta_0}^{\theta_0+\alpha} \big(f(z_0+re^{i\theta}) - f(z_0+r'e^{i\theta})\big)\, d\theta\right| \le \varepsilon.$$

It follows that F is also continuous on $[0, r_0]$. Consequently,

$$\lim_{r\to 0^+} F(r) = F(0) = \frac{1}{\alpha}\int_{\theta_0}^{\theta_0+\alpha} f(z_0)d\theta = f(z_0),$$

which is equivalent to (5.4.20). ■

The following simple consequence of Lemma 5.4.7 is useful.

Corollary 5.4.8. *Suppose that g is an analytic function in a deleted neighborhood of z_0 with a simple pole at z_0. For $0 < r \le r_0$, let σ_r be the circular arc at angle α (see Figure 5.29). Then we have*

$$\lim_{r\to 0^+} \int_{\sigma_r} g(z)\, dz = i\alpha \operatorname{Res}\big(g, z_0\big). \tag{5.4.21}$$

Proof. Let $f(z) = (z - z_0)g(z)$ for $z \ne z_0$ and define $f(z_0) = \lim_{z\to z_0}(z - z_0)g(z) = \operatorname{Res}(g, z_0)$. Since $g(z)$ has a simple pole at z_0, it follows from Theorem 4.5.15 that f is analytic at z_0. Now apply Lemma 5.4.7 to f and (5.4.21) follows from (5.4.20), since for $z \ne z_0$, $f(z)/(z - z_0) = g(z)$. ■

The improper integrals that we consider next differ from previous ones in that they are not always convergent; however, their Cauchy principal values do exist.