be (Figure 5.26)

$$\text{P.V.}\int_{-1}^{1} f(x)\,dx = \lim_{r\to 0^+}\left(\int_{-1}^{-r} f(x)\,dx + \int_{r}^{1} f(x)\,dx\right). \tag{5.4.17}$$

The Cauchy principal value of an integral may exist even though the integral itself is not convergent. For example,

$$\int_{-1}^{-r}\frac{dx}{x} + \int_{r}^{1}\frac{dx}{x} = 0$$

for all $r > 0$, so $\text{P.V.}\int_{-1}^{1}\frac{dx}{x} = 0$, but the integral itself is clearly not convergent since $\int_0^1 \frac{dx}{x} = \infty$. However, whenever $\int_{-1}^{1} f(x)\,dx$ is convergent the $\text{P.V.}\int_{-1}^{1} f(x)\,dx$ exists and the two integrals are the same, since we can split the limit of a sum in (5.4.17) into a sum of limits and recover (5.4.16).

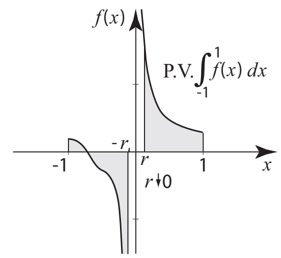

Fig. 5.26 The Cauchy principal value.

This fact allows us to compute convergent integrals by computing their principal values. We illustrate these ideas ~~in~~ via an example.

Example 5.4.6. (Cauchy principal values and singular points) Show that

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{1}{x-1}\,dx = 0 \tag{5.4.18}$$

by writing the integral in terms of limits of integrals over finite intervals.

Solution. The integral is improper as it extends over the infinite real line and the integrand is singular at $x = 1$. Accordingly, the principal value involves two limits

$$\begin{aligned}\text{P.V.}\int_{-\infty}^{\infty}\frac{1}{x-1}\,dx &= \lim_{R\to\infty}\left(\int_{-R}^{0}\frac{1}{x-1}\,dx + \int_{2}^{R}\frac{1}{x-1}\,dx\right) \\ &\quad + \lim_{r\to 0^+}\left(\int_{0}^{1-r}\frac{1}{x-1}\,dx + \int_{1+r}^{2}\frac{1}{x-1}\,dx\right),\end{aligned} \tag{5.4.19}$$

where the choices $x = 0$ and $x = 2$ were arbitrary; in fact, any pair of numbers with the first being in $(-\infty, 1)$ and the second in $(1, \infty)$ will work equally well in the splitting of the integrals. See Figure 5.27. Using elementary methods, we can see that both limits on the right of (5.4.19) exist, so the principal value exists, and we can write

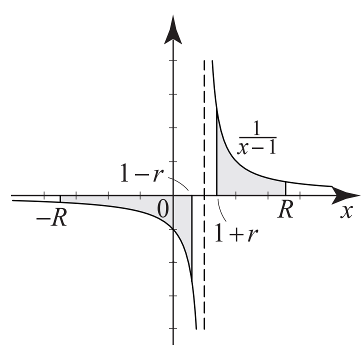

Fig. 5.27 The integral in (5.4.18).