5.4 Products of Rational and Trigonometric Functions

$$\frac{\pi}{e^a}i = \int_{-\infty}^{\infty} \frac{x}{x^2 + a^2} e^{ix} dx = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + a^2} dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx,$$

and the desired identity follows upon taking imaginary parts on both sides. Notice that the convergence of the improper integral in (5.4.14) follows by letting $R \to \infty$ in (5.4.15).

Indenting Contours

Taking a = 0 in Example 5.4.5, we obtain the formula

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi.$$

This identity is in fact needed in many applications. In the remainder of this section, we develop a method to calculate similar this and other interesting integrals.

In Section 5.3 we defined the Cauchy principal value of an improper integral over the real line. However, an integral can also be improper if the integrand becomes unbounded at a point inside the interval of integration. To make our discussion concrete, consider

$$\int_{-1}^{1} f(x) \, dx,$$

where *f* is a continuous function on [-1,0) and (0,1] but might have infinite limits as *x* approaches 0 from the left or right. Such an integral is said to be **convergent** if both $\lim_{b\to 0^-} \int_{-1}^{b} f(x) dx$ and $\lim_{a\to 0^+} \int_{a}^{1} f(x) dx$ are convergent. In this case, we set (Figure 5.25)



Fig. 5.25 Splitting an improper integral.

$$\int_{-1}^{1} f(x) \, dx = \lim_{b \to 0^{-}} \int_{-1}^{b} f(x) \, dx + \lim_{a \to 0^{+}} \int_{a}^{1} f(x) \, dx. \tag{5.4.16}$$

This expression should be contrasted with the one in which a function is integrated on intervals that approach the singular point x = 0 in a symmetric fashion. We define the Cauchy principal value of the integral $\int_{-1}^{1} f(x) dx$, with a singularity at x = 0, to