Corollary 5.4.4. *Let p, q ~~are~~ be complex polynomials with degree $q \geq 1 +$ degree p. Let σ_R denote the semi-circular arc consisting of all $z = Re^{i\theta}$, where $0 \leq \theta \leq \pi$. Then $\lim_{R\to\infty} \int_{\sigma_R} e^{isz} \frac{p(z)}{q(z)}\, dz = 0$ for all $s > 0$.*

Proof. Let $M(R)$ denote the maximum of $|p(z)/q(z)|$ for z on σ_R. Since degree $q \geq 1 +$ degree p, $M(R) \to 0$ as $R \to \infty$. Applying Lemma 5.4.3 we obtain the claimed assertion. ∎

Next we evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx.$$

It is not difficult to show that this integral is convergent using integration by parts (Exercise 19). However, because the degree of $x^2 + a^2$ is only one more than the degree of x, the estimate in Step 3 of Example 5.4.1 will not be sufficient to show that the integral on the expanding semi-circle tends to 0. For this purpose we appeal to Jordan's lemma.

Example 5.4.5. (Applying Jordan's lemma) Derive the identity

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \frac{\pi}{e^a}, \quad a > 0. \tag{5.4.14}$$

Solution. Consider the contour integral

$$I_{\gamma_R} = \int_{\sigma_R} \frac{z}{z^2 + a^2} e^{iz}\, dz + \int_{-R}^{R} \frac{x}{x^2 + a^2} e^{ix}\, dx = I_{\sigma_R} + I_R, \tag{5.4.15}$$

where σ_R is the circular arc shown in Figure 5.24 and $\gamma_R = \sigma_R \cup [-R, R]$. By Jordan's lemma (precisely by Corollary 5.4.4), $\lim_{R\to\infty} I_{\sigma_R} = 0$. For $R > a$, $\frac{z}{z^2+a^2} e^{iz}$ has a simple pole inside γ_R at ia. By the residue theorem, for all $R > a$, we obtain

$$\begin{aligned} I_{\gamma_R} &= 2\pi i \operatorname{Res}\left(\frac{z e^{iz}}{z^2 + a^2}, ia\right) \\ &= 2\pi i \frac{(ia) e^{i(ia)}}{2(ia)} \\ &= \frac{\pi}{e^a} i. \end{aligned}$$

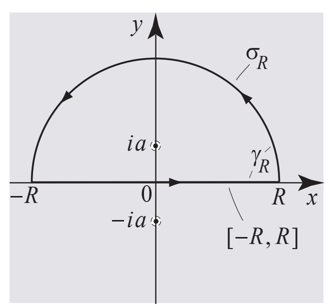

Fig. 5.24 The path and poles for the contour integral in Example 5.4.5.

Taking the limit as $R \to \infty$ in (5.4.15) and using the fact that $I_{\sigma_R} \to 0$, we deduce