$$2\int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}R\theta}\,d\theta = 2\int_0^R e^{-t}\,dt\,\frac{\pi}{2R} = 2(1-e^{-R})\frac{\pi}{2R} \le \frac{\pi}{R},$$

and combining this inequality with (5.4.12) concludes the proof of the lemma. ■

Lemma 5.4.3. (General Version of Jordan's Lemma) *Let $R_0 > 0$ and $0 \le \theta_1 < \theta_2 \le \pi$. For $R \ge R_0$, let σ_R be the circular arc of all $z = Re^{i\theta}$ with $0 \le \theta_1 \le \theta \le \theta_2 \le \pi$ as shown in Figure 5.23. Let f be a continuous complex-valued function defined on all arcs σ_R and let $M(R)$ denote the maximum value of $|f|$ on σ_R. If $\lim_{R\to\infty} M(R) = 0$, then for all $s > 0$*

$$\lim_{R\to\infty}\int_{\sigma_R} e^{isz} f(z)\,dz = 0.$$

Proof. For $s > 0$, we have from (5.4.8), $\left|e^{isz}\right| = e^{-sR\sin\theta}$.

Note that since $e^{-sR\sin\theta} > 0$, its integral increases if we increase the size of the interval of integration. Thus

$$\int_{\theta_1}^{\theta_2} e^{-sr\sin\theta}\,d\theta \le \int_0^{\pi} e^{-sr\sin\theta}\,d\theta,$$

if $0 \le \theta_1 \le \theta_2 \le \pi$. Parametrize σ_R by $\gamma(\theta) = Re^{i\theta}$, where $\theta_1 \le \theta \le \theta_2$. Then

$$\gamma'(\theta) = Rie^{i\theta}, \qquad |\gamma'(\theta)|d\theta = Rd\theta$$

and hence

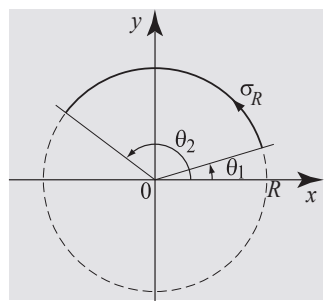

Fig. 5.23 The circular arcs σ_R

$$\begin{aligned}\left|\int_{\sigma_R} e^{isz} f(z)\,dz\right| &\le \int_{\theta_1}^{\theta_2} \left|e^{isz} f(z)\right| R\,d\theta \\ &\le RM(R)\int_{\theta_1}^{\theta_2} e^{-sR\sin\theta}\,d\theta \\ &\le RM(R)\int_0^{\pi} e^{-sR\sin\theta}\,d\theta. \qquad (5.4.13)\end{aligned}$$

From inequality (5.4.9) we obtain that the integral in (5.4.13) is bounded by $\pi/(sR)$. Thus

$$\left|\int_{\sigma_R} e^{isz} f(z)\,dz\right| \le RM(R)\frac{\pi}{sR} = \frac{\pi}{s}M(R) \to 0$$

as $R \to \infty$. This concludes the proof of the lemma. ■

An analog of Jordan's lemma holds for $s < 0$ if the circular arc σ_R is in the lower half-plane. Applying Jordan's lemma in the special case when $f(z)$ is a rational function, we obtain the following useful result.