$$\int_{-\infty}^{\infty} \frac{\cos(sx)}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-sa}. \qquad \square$$

By observing that $\sin\theta \ge 0$ for $0 \le \theta \le \pi$, we were able in (5.4.8) to obtain the inequality $\left|e^{isz}\right| \le 1$ for all $s \ge 0$ and $z = Re^{i\theta}$, $0 \le \theta \le \pi$. A more precise analysis of $\sin\theta$ yields a better estimate on $\left|e^{isz}\right|$, which in turn makes it possible to compute integrals of the form (5.4.1), where degree $q \ge 1 +$ degree $p$.

**Lemma 5.4.2. (Jordan's Lemma)** *The following inequality is valid for* $R > 0$*:*

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta \le \frac{\pi}{R}. \tag{5.4.9}$$

*Proof.* We begin by writing

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta = \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\,d\theta.$$

Then we change variables $t = \pi - \theta$ in the second integral above and, noting that $\sin(\theta) = \sin(\pi-\theta) = \sin t$ and $d\theta = -dt$, we obtain

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta = \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta - \int_{\frac{\pi}{2}}^{0} e^{-R\sin t}\,dt = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta. \tag{5.4.10}$$

At this point, we need an estimate on $\sin\theta$. On the interval $[0, \frac{\pi}{2}]$, the graph of $\sin\theta$ is concave down, because the second derivative is negative. Hence the graph of $\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$ is above the chordal line that joins two points on the graph. In particular, it is above the chord that joins the origin to the point $(\frac{\pi}{2}, 1)$, whose equation is $y = \frac{2}{\pi}\theta$.

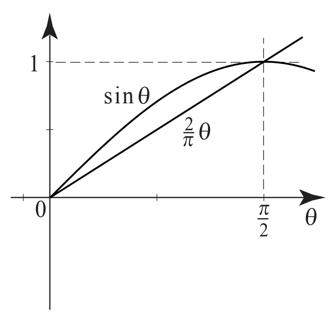


**Fig. 5.22** Proof of (5.4.11).

This fact is expressed analytically by the inequality

$$\sin\theta \ge \frac{2}{\pi}\theta, \tag{5.4.11}$$

which is valid for $0 \le \theta \le \frac{\pi}{2}$, whose geometric proof is illustrated in Figure 5.22.

Inequality (5.4.11) implies that $-\sin\theta \le -\frac{2}{\pi}\theta$ and combining this with (5.4.10) we deduce

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta \le 2\int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}R\theta}\,d\theta. \tag{5.4.12}$$

Changing variables $t = \frac{2}{\pi}R\theta$, we write