Step 2: Set up and evaluate the contour integral. Since

$$\int_{-\infty}^{\infty} \frac{\cos(sx)}{x^2+a^2}\,dx = \int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{e^{isx}}{x^2+a^2}\right)dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+a^2}\,dx\right), \tag{5.4.5}$$

we will consider the contour integral

$$I_{\gamma_R} = \int_{\gamma_R} \frac{e^{isz}}{z^2+a^2}\,dz = \int_{\sigma_R} \frac{e^{isz}}{z^2+a^2}\,dz + \int_{-R}^{R} \frac{e^{isx}}{x^2+a^2}\,dx = I_{\sigma_R} + I_R, \tag{5.4.6}$$

where γ_R and σ_R are as in Figure 5.21. For $R > a$, $\frac{e^{isz}}{z^2+a^2}$ has one simple pole inside γ_R at $z = ia$. In view of Proposition 5.1.3(ii), the residue there is

$$\operatorname{Res}\left(\frac{e^{isz}}{z^2+a^2}, ia\right) = \frac{e^{is(ia)}}{2ia} = \frac{e^{-sa}}{2ia}.$$

By the residue theorem, for all $R > a$, we have

$$I_{\gamma_R} = I_{\sigma_R} + I_R = 2\pi i \frac{e^{-sa}}{2ia} = \frac{\pi}{a} e^{-sa}. \tag{5.4.7}$$

Step 3: Show that $\lim_{R\to\infty} I_{\sigma_R} = 0$. For $s \geq 0$ and $0 \leq \theta \leq \pi$, we have $\sin\theta \geq 0$, hence $-sR\sin\theta \leq 0$, and so $e^{-sR\sin\theta} \leq 1$. Write z on σ_R, as $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$, where $0 \leq \theta \leq \pi$. Then

$$\left|e^{isz}\right| = \left|e^{isR(\cos\theta+i\sin\theta)}\right| = \overbrace{\left|e^{isR\cos\theta}\right|}^{1}\left|e^{-sR\sin\theta}\right| = e^{-sR\sin\theta} \leq 1. \tag{5.4.8}$$

Hence, for $R > a$ and z on the semi-circle σ_R, we have

$$\left|\frac{e^{isz}}{z^2+a^2}\right| \leq \frac{1}{|z^2+a^2|} \leq \frac{1}{|z|^2-a^2} = \frac{1}{R^2-a^2},$$

and so the *ML*-inequality for path integrals yields

$$\left|\int_{\sigma_R} \frac{e^{isz}}{z^2+a^2}\,dz\right| \leq \ell(\sigma_R)\,\frac{1}{R^2-a^2} = \frac{\pi R}{R^2-a^2} \to 0, \text{ as } R \to \infty.$$

This estimate works because the degree of the polynomial in the denominator is greater than the degree of the polynomial in the numerator by 2.

Step 4: Compute the desired improper integral. Let $R \to \infty$ in (5.4.7), use Step 3, and get

$$\lim_{R\to\infty} I_{\sigma_R} + \lim_{R\to\infty} I_R = 0 + \int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-sa}.$$

Taking real parts on both sides and using (5.4.5), we obtain the desired integral