Example 5.3.8. (An integral involving $\ln x$**)** Derive the identity

$$\int_0^\infty \frac{\ln x}{x^4+1}\,dx = -\frac{\pi^2}{8\sqrt{2}}. \tag{5.3.15}$$

Solution. Let $x = e^t$, $\ln x = t$, $dx = e^t\,dt$. This transforms the integral into

$$\int_{-\infty}^{\infty} \frac{t}{e^{4t}+1} e^t\,dt = \int_{-\infty}^{\infty} \frac{xe^x}{e^{4x}+1}\,dx.$$

To evaluate this integral, we integrate the function

$$f(z) = \frac{ze^z}{e^{4z}+1}$$

over the rectangular contour γ_R in Figure 5.20. Let I_j denote the integral of f over γ_j. Here again, we chose the vertical sides of γ_R so that on the returning path γ_3 the denominator equals to $e^{4x}+1$. As we will see momentarily, this enables us to relate I_3 to I_1. Let us now compute $I_{\gamma_R} = \int_{\gamma_R} f(z)\,dz$.

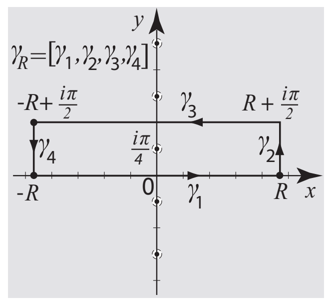

Fig. 5.20 The path and poles for the contour integral in Example 5.3.8.

The function f has one (simple) pole at $z = i\frac{\pi}{4}$ inside γ_R. By Proposition 5.1.3(ii), the residue there is

$$\operatorname{Res}\left(\frac{ze^z}{e^{4z}+1}, i\frac{\pi}{4}\right) = \frac{i\frac{\pi}{4}e^{i\frac{\pi}{4}}}{4e^{i\pi}} = -\frac{i\pi(1+i)}{16\sqrt{2}}.$$

So by the residue theorem

$$I_{\gamma_R} = 2\pi i\left(-\frac{i\pi(1+i)}{16\sqrt{2}}\right) = \frac{\pi^2(1+i)}{8\sqrt{2}} = I_1 + I_2 + I_3 + I_4. \tag{5.3.16}$$

Examining each I_j ($j = 1, \ldots, 4$), we have

$$I_1 = \int_{\gamma_1} \frac{ze^z}{e^{4z}+1}\,dz = \int_{-R}^{R} \frac{xe^x}{e^{4x}+1}\,dx.$$

On γ_3, $z = x + i\frac{\pi}{2}$, $dz = dx$, so using $e^{\frac{i\pi}{2}} = i$, we get