There are interesting integrals of rational functions that are not computable using semi-circular contours as in Example 5.3.3. One such integral is

$$\int_0^\infty \frac{dx}{x^3+1}. \tag{5.3.13}$$

This integral can be reduced to an integral involving exponential functions, via the substitution $x = e^t$. We outline this useful technique in the following example.

Example 5.3.7. (The substitution $x = e^t$**)** For $\alpha > 1$ establish the identity

$$\int_0^\infty \frac{1}{x^\alpha+1}\,dx = \frac{\pi}{\alpha \sin\frac{\pi}{\alpha}}. \tag{5.3.14}$$

Solution. Step 1: Show that the integral converges. The integrand is continuous, so it is enough to show that the integral converges on $[1,\infty)$. We have $\frac{1}{x^\alpha+1} \le \frac{1}{x^\alpha}$, and the integral is convergent since $\int_1^\infty \frac{1}{x^\alpha}\,dx < \infty$.

Step 2: Apply the substitution $x = e^t$. Let $x = e^t$, $dx = e^t\,dt$, and note that as x varies from 0 to ∞, t varies from $-\infty$ to ∞, and so

$$I = \int_0^\infty \frac{1}{x^\alpha+1}\,dx = \int_{-\infty}^\infty \frac{e^t}{e^{\alpha t}+1}\,dt = \int_{-\infty}^\infty \frac{e^x}{e^{\alpha x}+1}\,dx,$$

where, for convenience, in the last integral we have used x as a variable of integration instead of t. Identity (5.3.14) follows now from Example 5.3.6. □

The tricky part in Example 5.3.6 is choosing the contour. Let us clarify this part with one more example. For instance, we compute the integral

$$\int_0^\infty \frac{\ln x}{x^4+1}\,dx.$$

This integral is improper as the interval of integration is infinite and the integrand tends to $-\infty$ as $x \downarrow 0$. To define the convergence of such integrals, we follow the general procedure of taking all one-sided limits one at a time; see Figure 5.19. Thus

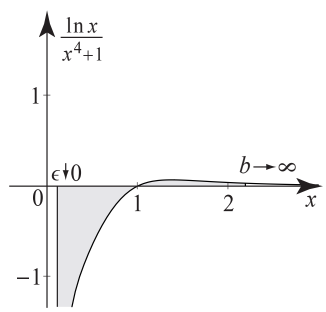

Fig. 5.19 Splitting an improper integral.

$$\int_0^\infty \frac{\ln x}{x^4+1}\,dx = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 \frac{\ln x}{x^4+1}\,dx + \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^4+1}\,dx.$$

It is not difficult to show that both limits exist and thus the integral converges. We ~~look at~~ consider this integral in the next example.