Furthermore, the series can be integrated term by term over any path that lies in the annulus $0<\left|z-z_{0}\right|<R$. Let $C_{r}\left(z_{0}\right)$ be a positively oriented circle that lies in $0<$ $\left|z-z_{0}\right|<R$. If we integrate the Laurent series term by term over $C_{r}\left(z_{0}\right)$ and use the fact that $\int_{C_{r}\left(z_{0}\right)}\left(z-z_{0}\right)^{n} d z=0$ if $n \neq-1$ and $\int_{C_{r}\left(z_{0}\right)} \frac{1}{z-z_{0}} d z=2 \pi i$ (Example 3.2.10), we find $\int_{C_{r}\left(z_{0}\right)} f(z) d z=a_{-1} 2 \pi i$; hence

$$
\begin{equation*}
a_{-1}=\frac{1}{2 \pi i} \int_{C_{r}\left(z_{0}\right)} f(z) d z \tag{5.1.1}
\end{equation*}
$$

Definition 5.1.1. The coefficient $a_{-1}$ in (5.1.1) is called the residue of $f$ at $z_{0}$ and is denoted by $\operatorname{Res}\left(f, z_{0}\right)$ or simply $\operatorname{Res}\left(z_{0}\right)$ when there is no risk of confusion.

With the concept of residue in hand, we state our main result of this section which reduces the evaluation of certain integrals to computations of residues.

Theorem 5.1.2. (Cauchy's Residue Theorem) Let C be a simple closed positively oriented path. Suppose that $f$ is analytic inside and on $C$, except at finitely many isolated singularities $z_{1}, z_{2}, \ldots, z_{n}$ inside $C$. Then

$$
\begin{equation*}
\int_{C} f(z) d z=2 \pi i \sum_{j=1}^{n} \operatorname{Res}\left(f, z_{j}\right) \tag{5.1.2}
\end{equation*}
$$

Proof. We start by picking small circles $C_{r_{j}}\left(z_{j}\right)(j=1,2, \ldots, n)$ that do not intersect each other and are contained in the interior of $C$ (Figure 5.1). Apply Cauchy's integral theorem for multiple simple paths multiply connected regions (Theorem 3.7.2) to obtain
$\int_{C} f(z) d z=\sum_{j=1}^{n} \int_{C_{r_{j}}} f(z) d z=2 \pi i \sum_{j=1}^{n} \operatorname{Res}\left(z_{j}\right)$,
where the last equality follows from (5.1.1). So (5.1.2) holds.


Fig. 5.1 The path $C$ and the circles $C_{r_{j}}\left(z_{j}\right)$.

The computation of the residue will depend on the type of singularity of the function $f$, as we now illustrate.

Proposition 5.1.3. (i) Suppose that $z_{0}$ is an isolated singularity of $f$. Then $f$ has a simple pole at $z_{0}$ if and only if

$$
\begin{equation*}
\operatorname{Res}\left(f, z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right) f(z) \neq 0 . \tag{5.1.3}
\end{equation*}
$$

(ii) If $f(z)=\frac{p(z)}{q(z)}$, where $p$ and $q$ are analytic at $z_{0}, p\left(z_{0}\right) \neq 0$, and $q$ has a simple zero at $z_{0}$, then

