Proof. Let f be as in the statement of the theorem. Set $b = f(a)$ and consider the function

$$F = \phi_b \circ f \circ \phi_{-a}$$

defined on the unit disc $B_1(0)$; see Figure 4.19. Then F maps $B_1(0)$ to $\overline{B_1(0)}$ and we have

$$F(0) = \phi_b \circ f \circ \phi_{-a}(0) = \phi_b(f(a)) = 0\,,$$

and moreover $|F(z)| \le 1$ for all $|z| < 1$. It follows from Lemma 4.6.1 that

$$|F(w)| = |\phi_b \circ f \circ \phi_{-a}(w)| \le |w|,$$

so letting $\phi_{-a}(w) = z$ and $\phi_a(z) = w$, we get $|\phi_b \circ f(z)| \le |\phi_a(z)|$ for $z \in B_1(0)$. Using the definition of ϕ_b we deduce (4.6.5).

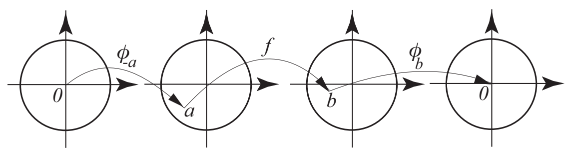

Fig. 4.19 The function $F = \phi_b \circ f \circ \phi_{-a}$.

Now consider $F'(0) = (\phi_b \circ f \circ \phi_{-a})'(0)$. Applying the chain rule, we write

$$F'(0) = \phi_b'(f(\phi_{-a}(0)) f'(\phi_{-a}(0)) \phi_{-a}'(0).$$

Since $\phi_{-a}'(0) = 1 - |a|^2$ and $\phi_{-a}(0) = a$ we obtain

$$(\phi_b \circ f \circ \phi_{-a})'(0) = \frac{1-|b|^2}{(1-|b|^2)^2} f'(a)(1-|a|^2) = f'(a)\frac{1-|a|^2}{1-|b|^2}\,.$$

By Lemma 4.6.1 this number should have modulus at most 1, thus we obtain (4.6.6).

Suppose now that equality holds in (4.6.5) for some $z \ne a$ or that equality holds in (4.6.6). Then by Lemma 4.6.1 we must have $\phi_b \circ f \circ \phi_{-a}(z) = cz$ for $|c| = 1$, so then $f \circ \phi_{-a}(z) = \phi_{-b}(cz)$ and leting $\phi_{-a}(z) = w$, we see that $f(w) = \phi_{-b}(c\phi_a(w))$. Using Proposition 4.6.2 we write

$$f(w) = \phi_{-b}(c\phi_a(w)) = c\,\phi_{-b\bar c}(\phi_a(w)) = c\tfrac{1-b\bar a\bar c}{1-\bar b a c}\,\phi_{\frac{a-b\bar c}{1-b\bar a\bar c}}(z) = d\,\phi_q(z)\,,$$

with

$$d = c\tfrac{1-b\bar a\bar c}{1-\bar b a c}, \qquad q = -\phi_a(b\bar c)\,, \quad \text{(note } |d| = 1,\ |q| < 1\text{)},$$

thus f must be a Möbius transformation times a unimodular constant. ■

Example 4.6.4. Let f be an analytic map from the unit disc to itself.
(a) If f fixes a point a in the unit disc, i.e., $f(a) = a$, show that $|f'(a)| \le 1$.
(b) Show that $|f(0)|^2 + |f'(0)| \le 1$.