and this z satisfies $|z| < 1$ if $|w| < 1$ and $|z| = 1$ if $|w| = 1$. This also proves (iv).

To prove (v) we use the quotient rule to write

$$\phi_a'(z) = \frac{(1-\overline{a}z)-(z-a)(-\overline{a})}{(1-\overline{a}z)^2} = \frac{1-|a|^2}{(1-\overline{a}z)^2}.$$

To prove (vi) we write

$$\phi_a(cz) = \frac{cz-a}{1-\overline{a}cz} = \frac{cz-c\overline{c}a}{1-\overline{a}cz} = c\frac{z-a\overline{c}}{1-\overline{a\overline{c}}z} = c\,\phi_{a\overline{c}}(z).$$

Finally, (vii) is proved by a direct calculation as follows:

$$\phi_a \circ \phi_b(z) = \frac{(1+a\overline{b})z-(a+b)}{(1+\overline{a}b)-(\overline{a}+\overline{b})z} = \frac{1+a\overline{b}}{1+\overline{a}b}\,\frac{z-\dfrac{a+b}{1+a\overline{b}}}{1-\dfrac{\overline{a}+\overline{b}}{1+\overline{a}b}z} = \frac{1+a\overline{b}}{1+\overline{a}b}\,\phi_{\frac{a+b}{1+a\overline{b}}}(z).$$

Notice that the constant $\frac{1+a\overline{b}}{1+\overline{a}b}$ is unimodular and that $\frac{a+b}{1+a\overline{b}} = \phi_{-b}(a)$. ■

Let us examine how the properties of ϕ_a interplay with the maximum and minimum modulus principles. Clearly, ϕ_a is not constant, but we showed that $|\phi_a(z)|$ equals 1 on $C_1(0)$. Studying Corollary 3.9.10, we conclude that $\phi_a(z)$ must vanish somewhere inside the unit disk. This is certainly the case since $\phi_a(a) = 0$; and in fact $z = a$ is the only zero of $\phi_a(z)$ inside the unit disk.

The Schwarz-Pick theorem

Let f be a map defined on the unit disc $B_0(1)$ that satisfies $|f(z)| \le 1$ for all $|z| < 1$. We cannot directly apply Schwarz's Lemma on f since we may not have $f(0) = 0$. But we can compose f with two linear fractional transformations to achieve this.

Theorem 4.6.3. (Schwarz-Pick Theorem) *Let f be an analytic function that maps the open unit disc $B_1(0)$ to the closed unit disk $\overline{B_1(0)}$. Then for a number a in the disc $B_1(0)$ we have*

$$\left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \le \left|\frac{z-a}{1-\overline{a}z}\right| \tag{4.6.5}$$

and also

$$|f'(a)| \le \frac{1-|f(a)|^2}{1-|a|^2}. \tag{4.6.6}$$

Moreover, if equality holds in (4.6.5) *for some $z \ne a$ or equality holds in* (4.6.6), *then f is equal to a unimodular constant times a Möbius transformation.*