Möbius Transformations on the Unit Disk

Maps of the form $\frac{az+b}{cz+d}$ are called **linear fractional transformations**. We would like to study a class of linear fractional transformations from the open unit disk to itself of the form

$$\phi_a(z) = \frac{z-a}{1-\overline{a}\,z}, \qquad |z| \le 1 \tag{4.6.4}$$

where a is a complex number with $|a|<1$. As the denominator in (4.6.4) never vanishes when $|z|<1$, we note that these maps are analytic on the unit disc. A map ϕ_a is called a **Möbius transformation** after the mathematician August Ferdinand Möbius (1790–1868).

Proposition 4.6.2. (Properties of Möbius Transformations) *Let a,b be complex numbers with $|a|,|b|<1$. Then for all $|z|<1$ we have*

(i) $\phi_a(0) = -a$ *and* $\phi_a(a) = 0$.
(ii) ϕ_a *is a one-to-one and onto map from* $B_1(0)$ *to* $B_1(0)$.
(iii) ϕ_a *is a one-to-one and onto map from* $C_1(0)$ *to* $C_1(0)$.
(iv) $\phi_a^{-1} = \phi_{-a}$.
(v) $\phi_a'(z) = \dfrac{1-|a|^2}{(1-\overline{a}z)^2}$.
(vi) $\phi_a(c\,z) = c\,\phi_{a\overline{c}}(z)$ *where c is a complex number with* $|c|=1$.
(vii) We have the identity

$$\phi_a \circ \phi_b(z) = \frac{1+a\overline{b}}{1+\overline{a}\,b}\,\phi_{\frac{a+b}{1+a\overline{b}}}(z) = \frac{1+a\overline{b}}{1+\overline{a}\,b}\,\phi_{\phi_{-b}(a)}(z)\,.$$

Proof. Assertion (i) is trivial. To prove (ii) and (iii) notice that for θ real we have

$$|\phi_a(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-\overline{a}e^{i\theta}}\right| = \left|\frac{1-ae^{-i\theta}}{1-\overline{a}e^{i\theta}}\right| = 1.$$

This shows that ϕ_a maps $C_1(0)$ to $C_1(0)$. It follows that ϕ_a maps $B_1(0)$ to $B_1(0)$, otherwise it would achieve its maximum in the interior of $B_1(0)$, contradicting the maximum modulus principle (Theorem 3.9.6).

To prove the one-to-one assertion in (ii) and (iii) we assume $\frac{z-a}{1-\overline{a}z} = \frac{z'-a}{1-\overline{a}z'}$ and we show that $z=z'$. Cross multiplying gives $(z-a)(1-\overline{a}z') = (z'-a)(1-\overline{a}\,z)$ which implies $z-a-\overline{a}\,z'\,z+|a|^2z' = z'-a-\overline{a}\,z\,z'+|a|^2z$. This in turn implies $z-z' = |a|^2(z-z')$, hence $(z-z')(1-|a|^2)=0$, and so $z=z'$.

We prove the onto assertion in (ii) and (iii) by solving the equation $\frac{z-a}{1-\overline{a}z} = w$ in z for a given w in the unit disk or on the unit circle. Indeed, algebraic manipulations yield

$$z = \frac{w+a}{1+\overline{a}w} = \phi_{-a}(w)$$