(d) If $f(0) = 0$, show that $f(z) = az^n$, where n is the order of the zero at 0. [Hint: Factor z^n and apply (c) to the entire function $\frac{f(z)}{z^n}$.]

4.6 Schwarz's Lemma

In this section we put together some of the material we have developed to prove a very elegant and useful lemma attributed to Karl Hermann Amandus Schwarz (1843-1921). This lemma reflects the rigidity of analytic functions from the unit disk to itself and has remarkable applications.

Lemma 4.6.1. (Schwarz's Lemma) *Suppose that f is analytic on the open unit disk $B_1(0)$ with $f(0) = 0$ and that f takes values in the closed unit disk $\overline{B_1(0)}$, i.e., it satisfies $|f(z)| \le 1$ for all $|z| < 1$. Then we have*

$$|f(z)| \le |z| \quad \text{for all } |z| < 1 \tag{4.6.1}$$

and

$$|f'(0)| \le 1. \tag{4.6.2}$$

Moreover, if either (a) equality holds in (4.6.1) *for some $z \neq 0$ (i.e., there is a $z_0 \neq 0$ with $|z_0| < 1$ such that $|f(z_0)| = |z_0|$) or (b) equality holds in* (4.6.2)*, then there is a complex constant A with $|A| = 1$ such that*

$$f(z) = Az$$

for all $|z| < 1$.

Proof. Notice that $\lim_{w\to 0} \frac{f(w)}{w} = f'(0)$. Consider the function

$$g(w) = \begin{cases} \frac{f(w)}{w} & \text{if } w \neq 0, \\ f'(0) & \text{if } w = 0. \end{cases} \tag{4.6.3}$$

It follows from Theorem 4.5.12(iii) that g has a removable singularity and hence is analytic on the open unit disk. Fix a point z in $B_1(0)$ and let R satisfy $|z| < R < 1$. Then the function g is analytic on the open disk $B_R(0)$ and is continuous on its boundary $C_R(0)$; moreover $|g(w)| \le 1/R$ on the circle $|w| = R$ in view of the fact that $|f(w)| \le 1$ for all w in $B_1(0)$. It follows from the maximum modulus principle (Theorem 3.9.6) that $|g(w)| \le 1/R$ for all $|w| \le R$, in particular $|g(0)| \le 1/R$ and $|g(z)| \le 1/R$. Letting $R \uparrow 1$ we obtain $|g(0)| \le 1$ and $|g(z)| \le 1$. Recasting these in terms of f, we obtain $|f'(0)| \le 1$ and $|f(z)| \le |z|$. Since $|z| < 1$ was an arbitrary point in the unit disk, (4.6.1) and (4.6.2) hold.

Now suppose that equality holds in (4.6.1) for some $z = z_0 \neq 0$ or equality holds in (4.6.2). Then g attains its maximum in the interior of $B_1(0)$ and it must be equal to a constant A by Theorem 3.9.6. Thus $f(w) = Aw$ for all $|w| < 1$. Since there is a $z_0 \neq 0$ in $B_1(0)$ such that $|f(z_0)| = |z_0|$ or $f'(0) = 1$, it follows that $|A| = 1$. ■