denote the Laurent series expansion of f about z_0, which is valid in some annulus $0 < |z - z_0| < R$. Then
(i) z_0 is a removable singularity $\Leftrightarrow$ $a_n = 0$ for all $n = -1, -2, \dots$.
(ii) z_0 is a pole of order $m \geq 1$ $\Leftrightarrow$ $a_{-m} \neq 0$ for some $m > 0$ and $a_n = 0$ for all $n < -m$.
(iii) z_0 is an essential singularity $\Leftrightarrow$ $a_n \neq 0$ for infinitely many $n < 0$.

Example 4.5.18. (Essential singularities) Classify the isolated singularities of the function $f(z) = e^{\frac{z}{\sin z}}$.

Solution. The function f is analytic at all points except where $\sin z = 0$; that is, except when $z = k\pi$, where k is an integer. As it is a bit complicated to find the Laurent series expansion of f, we use the characterizations of Theorems 4.5.12 and 4.5.15. When $z = 0$, we have $\lim_{z\to 0} \frac{z}{\sin z} = 1$, hence $\lim_{z\to 0} e^{\frac{z}{\sin z}} = e$, and so the function has a removable singularity at $z = 0$ by Theorem 4.5.12(iii). We claim that we have an essential singularity at $z = k\pi$, $k \neq 0$. To prove this, we eliminate the possibility of a removable singularity or a pole. Suppose that $k > 0$ is even. Then it is easy to see that if $z = x$ is real, then $\lim_{x\downarrow k\pi} \frac{x}{\sin x} = +\infty$ and $\lim_{x\uparrow k\pi} \frac{x}{\sin x} = -\infty$. So if $z = x$ is real, then

$$\lim_{x\downarrow k\pi} e^{\frac{x}{\sin x}} = \infty,$$

implying that $k\pi$ is not a removable singularity of $e^{\frac{z}{\sin z}}$. Also,

$$\lim_{x\uparrow k\pi} e^{\frac{x}{\sin x}} = 0,$$

implying that $k\pi$ is not a pole of $e^{\frac{z}{\sin z}}$. This leaves only one possibility: $k\pi$ is an essential singularity of $e^{\frac{z}{\sin z}}$. A similar argument works for $k < 0$ even and odd k. $\square$

One way to determine whether an isolated singularity is an essential singularity is to rule out the possibility of the other two types of singularities. This can be achieved by showing that the function is unbounded and has different limits as we approach the isolated singularity in different ways. The following theorem is a useful characterization of essential singularities. The theorem was discovered independently by Weierstrass and the Italian mathematician Felice Casorati (1835–1890).

Theorem 4.5.19. (Casorati-Weierstrass) *Suppose that f is analytic on a punctured disk $B'_R(z_0)$. Then z_0 is an essential singularity of f if and only if the following two conditions hold:*
(i) There is a sequence $\{z_n\}_{n=1}^{\infty}$ in $B'_R(z_0)$ such that $z_n \to z_0$ and $|f(z_n)| \to \infty$ as $n \to \infty$.
(ii) For any complex number α, there is a sequence $\{z_n\}_{n=1}^{\infty}$ in $B'_R(z_0)$ (that depends on α) such that $z_n \to z_0$ and $f(z_n) \to \alpha$ as $n \to \infty$.

Remark 4.5.20. Notice that (i) is not saying that $\lim_{z\to z_0} |f(z)| = \infty$. It is just saying that you can approach z_0 in such a way that $|f(z)|$ will tend to infinity. Similarly, part